

17) Odrediti stacionarne tačke funkcije z

a) $z = 8xy^2 - xy^3 - 5x^2y$

b) $x^2 + 2y^2 + 2z^2 + 8xz - x + 8 = 0$

a) $z'_x = 8y^2 - y^3 - 10xy = 0$

$z'_y = 16xy - 3xy^2 - 5x^2 = 0$

$y(-y^2 + 8y - 10x) = 0$

$x(-5x^2 - 3y^2 + 16y) = 0$

1° $x = 0, y = 0 \quad (0, 0)$

2° $y = 0 \quad -5x^2 - 3y^2 + 16y = 0$
 $-5x^2 = 0 \Rightarrow x = 0$
 $(0, 0)$

3° $x = 0 \quad -y^2 + 8y - 10x^2 = 0$
 $-y(y - 8) = 0$
 $y = 0 \quad (0, 0)$
 $y = 8 \quad (0, 8)$

4° $-y^2 + 8y - 10x^2 = 0$
 $-5x^2 - 3y^2 + 16y = 0 \Rightarrow 5x^2 = -3y^2 + 16y$

$-y^2 + 8y - 3(-3y^2 + 16y) = 0$

$-y^2 + 8y + 9y^2 - 48y = 0$
 $8y^2 - 40y = 0 \quad | : 8$
 $y^2 - 5y = 0$

$y(y - 5) = 0$

$y = 0 \quad x = 0$
 $y = 5 \quad 5x^2 = -9.5 + 80 = 5$

$x^2 = 1$

$x = \pm 1$

$(1, 5)$
 $(-1, 5)$

~ IZVOD IMPLICITNO ZADATE F-KE ~

Kažemo da je jednačinom $F(x, y, z) = 0$ funkcija z koja zavisi od x i y zadata IMPLICITNO.

$dF = F_x dx + F_y dy + F_z dz, \quad dF = 0$

$z = f(x, y), \quad dz = f_x dx + f_y dy$

$0 = F_x dx + F_y dy + F_z (f_x dx + f_y dy)$

$\Leftrightarrow 0 = (F_x + F_z f_x) dx + (F_y + F_z f_y) dy$

$\Leftrightarrow 0 = F_x + F_z f_x \quad \wedge \quad 0 = F_y + F_z f_y$

$\Rightarrow \quad z'_x = \frac{\partial f}{\partial x} = - \frac{F_x}{F_z}, \quad z'_y = \frac{\partial f}{\partial y} = - \frac{F_y}{F_z}$

b) stacionarne tačke implicitno zadate funkcije mogu se dobiti rešavanjem

$F_x = 0 \quad \wedge \quad F_y = 0$

$x^2 + 2y^2 + 2z^2 + 8xz - x + 8 = 0$

$F_x = 2x + 8z - 1 = 0$

$8z = 1 - 2x$

$z = \frac{1-2x}{8}$

$F_y = 4y = 0$

$y = 0$

$$x^2 + 2y^2 + 2z^2 + 8xz - x + 8 = 0$$

$$x^2 + 0 + 2 \cdot \frac{1}{64} (1 - 4x + 4x^2) + 8x \cdot \frac{1-2x}{8} - x + 8 = 0$$

$$\textcircled{x^2} + \frac{1}{32} (1 - 4x + 4x^2) + x \textcircled{-2x^2} - x + 8 = 0 \quad / \cdot 32$$

$$-32x^2 + 1 - 4x + 4x^2 + 8 \cdot 32 = 0$$

$$28x^2 + 4x - 257 = 0$$

$$x_{1,2} = \frac{-4 \pm \sqrt{16 + 28784}}{2 \cdot 28} = \frac{-4 \pm \sqrt{28800}}{56} = \frac{-4 \pm \sqrt{400 \cdot 9 \cdot 8}}{56}$$

$$= \frac{-4 \pm \sqrt{400 \cdot 9 \cdot 4 \cdot 2}}{56} = \frac{-4 \pm 120\sqrt{2}}{56} = \frac{4(-1 \pm 30\sqrt{2})}{4 \cdot 14} = \frac{-1 \pm 30\sqrt{2}}{14}$$

stacionarne tačke su: $\left(\frac{-1+30\sqrt{2}}{14}, 0\right)$ i $\left(\frac{-1-30\sqrt{2}}{14}, 0\right)$

c) $f(x, y, z) = x^2 - xz + y^3 + z$

$$f_x = 2x - z = 0 \quad \Rightarrow \quad z = 2x \quad \begin{matrix} \nearrow \\ f = 2 \end{matrix}$$

$$f_y = 3y^2 = 0 \quad \Rightarrow \quad y = 0$$

$$f_z = -x + 1 = 0 \quad \Rightarrow \quad x = 1$$

stacionarna tačka je $(1, 0, 2)$

d) $f(x, y, z) = x^2 + \sin y + z^2$

$$f_x = 2x = 0 \quad \Rightarrow \quad x = 0$$

$$f_y = \cos y = 0 \quad \Rightarrow \quad y = \frac{\pi}{2} + k\pi, \quad k \in \mathbb{Z}$$

$$f_z = 2z = 0 \quad \Rightarrow \quad z = 0$$

$\Rightarrow$ imamo prebrojivo mnogo stacionarnih tačaka

$$T_k \left(0, \frac{\pi}{2} + k\pi, 0 \right), \quad k \in \mathbb{Z}$$

e) $x^2 + y^2 + z^2 - 2x + 2y - 4z - 10 = 0$

$$z_x = -\frac{F_x}{F_z}, \quad z_y = -\frac{F_y}{F_z}$$

stacionarne tačke dobijamo rešavanjem

$$F_x = 0$$

$$F_y = 0$$

$$F_x = 2x - 2 \quad \Rightarrow \quad x = 1$$

$$F_y = 2y + 2 \quad \Rightarrow \quad y = -1$$

$$(1, -1)$$

ZADATAK 12.

$$f) x^2 + y^2 + z^2 = 25$$

$$F(x, y, z) = x^2 + y^2 + z^2 - 25 = 0$$

$$z_x, z_y = ?$$

$$z_x = - \frac{F_x}{F_z}$$

$$z_y = - \frac{F_y}{F_z}$$

$$F_x = 2x$$

$$z_x = - \frac{2x}{2z} = - \frac{x}{z} \equiv 0 \Rightarrow x = 0$$

$$F_y = 2y$$

$$z_y = - \frac{2y}{2z} = - \frac{y}{z} \equiv 0 \Rightarrow y = 0$$

$$F_z = 2z$$

$$(0, 0)$$

$$g) z \ln z = x + y^2$$

$$z_x, z_y = ?$$

$$F = z \ln z - x - y^2 = 0$$

$$F_x = -1, \quad F_y = -2y, \quad F_z = \ln z + z \cdot \frac{1}{z} = \ln z + 1$$

$$z_x = - \frac{F_x}{F_z} = \frac{1}{\ln z + 1} \neq 0 \quad \text{nvek}$$

$$z_y = - \frac{F_y}{F_z} = + \frac{2y}{\ln z + 1} \equiv 0 \Rightarrow y = 0$$

} $\Rightarrow$ nema
stacionarnih
tačaka

18) Ispitati lokalne ekstremne vrednosti funkcije

a) $z = 4(x-y) - x^2 - y^2$

b) $z = x^3 + y^3 - 3xy$

c) $z = x^2y^2 - 2x^2y - 2xy^2 + 4xy$ ← domaći

d) $u = x + \frac{y^2}{4x} + \frac{z^2}{y} + \frac{2}{x}$, $x, y, z > 0$ ← domaći

e) $f(x, y, z) = x^2 - xz + y^3 + z$

$f = f(x, y)$

$df(x, y) = f_x dx + f_y dy$

$d^2f(x, y) = f_{xx}(dx)^2 + f_{xy} dx dy + f_{yx}(dy)^2 + f_{yx} dy dx$

a) $z_x = 4(1-0) - 2x - 0 = 4 - 2x \equiv 0$

$2x = 4$ $x = 2$

$z_y = 4(0-1) - 0 - 2y = -4 - 2y \equiv 0$

$2y = -4$ $y = -2$

stacionarna tačka je $(2, -2)$

$z_{xx} = -2$

$z_{yy} = -2$

$z_{xy} = 0$

$z_{yx} = 0$

$d^2z = -2(dx)^2 - 2(dy)^2 = -2((dx)^2 + (dy)^2) < 0$

⇒ funkcija ima maksimum
u $(2, -2)$

$$b) \quad z = x^3 + y^3 - 3xy$$

$$z_x = 3x^2 - 3y$$

$$z_{xy} = -3$$

$$z_{xx} = 6x$$

$$z_y = 3y^2 - 3x$$

$$z_{yx} = -3$$

$$z_{yy} = 6y$$

$$\begin{aligned} 3x^2 - 3y &= 0 \\ 3y^2 - 3x &= 0 \end{aligned}$$

$$\Rightarrow y = x^2$$

$$3x^4 - 3x = 0$$

$$3x(x^3 - 1) = 0$$

$$x = 0 \quad \vee \quad x^3 - 1 = 0$$

$$x^3 = 1 \Rightarrow x = 1$$

$$\downarrow$$

$$y = 0$$

$$\downarrow$$

$$y = 1$$

stacionarne tačke: $(0,0)$ i $(1,1)$

$$d^2f(x,y) = 6x(dx)^2 - 6dxdy + 6y(dy)^2$$

$$d^2f(0,0) = -6dxdy$$

→ menja znak u zavisnosti od dx i dy

⇒ nema ni min ni max u $(0,0)$

$$d^2f(1,1) = 6(dx)^2 - 6dxdy + 6(dy)^2 = 6((dx)^2 - dxdy + (dy)^2)$$

$$= 6\left(\left(dx - \frac{dy}{2}\right)^2 + \frac{3}{4}(dy)^2\right) > 0$$

→ funkcija ima minimum u $(1,1)$

$$e) \quad f(x,y,z) = x^2 - xz + y^3 + z$$

$$f_x = 2x - z = 0 \Rightarrow 2x = z \quad z = 2$$

$$f_y = 3y^2 = 0 \Rightarrow y = 0$$

$$f_z = -x + 1 = 0 \Rightarrow x = 1$$

stacionarna tačka $(1,0,2)$

$$f_{xx} = 2 \quad f_{xy} = 0 \quad f_{xz} = -1$$

$$f_{yy} = 6y \quad f_{yx} = 0 \quad f_{yz} = 0$$

$$f_{zz} = 0 \quad f_{zx} = -1 \quad f_{zy} = 0$$

$$d^2f(x,y,z) = 2(dx)^2 - dxdz + 6y(dy)^2 - dzdx$$

$$= 2(dx)^2 - 2dxdz + 6y(dy)^2$$

$$= 2(dx)^2 - 2dxdz = 2dx(dx - dz) \xrightarrow{y=0} \text{ovo menja znak!}$$

f-ja nema ni min ni max u $(1,0,2)$

19) Odrediti lokalne ekstreumne vrednosti funkcije

a) $f(x,y) = x^2 + xy + y^2$, uz uslov $4x^2 + 4xy + y^2 = 1$

b) $f(x,y) = 2x^2 + 4xy - y^2$, uz uslov $x^2 + y^2 = 1$

c) $f(x,y,z) = xy + yz$, uz uslov $x + y = 2$, $y + z = 2$

d) $f(x,y) = xy$, uz uslov $x + y = 1$

prvo rešavam d) $\rightarrow$ on je najlakši :)

uslov $g_1(x,y) = x + y - 1 = 0$

formiramo Lagrange-ovu f-jm $\Phi(x,y;\lambda) = xy + \lambda(x + y - 1)$

Sada tražim stacionarnu tačku :

$$\left. \begin{aligned} \Phi_x(x,y;\lambda) &= y + \lambda = 0 \\ \Phi_y(x,y;\lambda) &= x + \lambda = 0 \\ g_1(x,y) &= x + y - 1 = 0 \end{aligned} \right\} \text{ rešavamo sistem}$$

$$\begin{aligned} y &= -\lambda \\ x &= -\lambda \\ -2\lambda - 1 &= 0 \end{aligned}$$

$$\Rightarrow \begin{aligned} y &= 1/2 \\ x &= 1/2 \\ \lambda &= -1/2 \end{aligned}$$

$N_\Phi(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ - stacionarna tačka f-je Φ

$N_f(\frac{1}{2}, \frac{1}{2})$ - stacionarna tačka f-je f

$$d^2\Phi(x,y;\lambda) = \Phi_{xx}(dx)^2 + \Phi_{xy}dx dy + \Phi_{yx}dy dx + \Phi_{yy}(dy)^2$$

$$\Phi_{xx} = 0 \quad \Phi_{xy} = 1$$

$$\Phi_{yx} = 1 \quad \Phi_{yy} = 0$$

$$d^2\phi(x,y;\lambda) = dx dy + dy dx = 2dx dy$$

Sada diferenciramo uslov : $x+y=1$

$$dx+dy=0 \Rightarrow dx=-dy, dy \neq 0$$

(jer ako bi bilo $dy=0 \Rightarrow dx=0$ y bar jedan $d_i \neq 0$)

pošto je $d^2\phi(x,y;\lambda) = -2(dy)^2 < 0 \Rightarrow$ funkcija ima maksimum

$$\text{uslovni maksimum je } f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$g(x,y) = x^2 + xy + y^2$$

$$g_1(x,y) = 4x^2 + 4xy + 4y^2 - 1 = 0$$

$$\phi(x,y,\lambda) = x^2 + xy + y^2 + \lambda(4x^2 + 4xy + 4y^2 - 1) \quad \text{Lagrange-ova f-ja}$$

$$\phi_x(x,y,\lambda) = 2x + y + 4\lambda x + 4\lambda y = x(2+4\lambda) + y(1+4\lambda)$$

$$\phi_y(x,y,\lambda) = x + 2y + 4\lambda x + 4\lambda y = (1+4\lambda)x + y(2+4\lambda)$$

$$g_1(x,y) = 4x^2 + 4xy + 4y^2 - 1 = 0$$

$$(1+4\lambda)(2x+y) = 0 \Leftrightarrow 1+4\lambda = 0 \vee 2x+y=0 \Leftrightarrow \lambda = -\frac{1}{4} \vee y = -2x$$

$$(1+4\lambda)(x+2y) = 0 \Leftrightarrow 1+4\lambda = 0 \vee x+2y=0 \Leftrightarrow \lambda = -\frac{1}{4} \vee x = -2y$$

$$4x^2 + 4xy + 4y^2 - 1 = 0$$

$$\lambda = -\frac{1}{4}$$

$$4x^2 + 4xy + 4y^2 - 1 = 0$$

$$a) f(x,y) = x^2 + xy + y^2$$

$$\Leftrightarrow (4x+2y)(2dx+dy) = 0$$

$$g(x,y) = 4x^2 + 4xy + y^2 - 1 = 0$$

$$\Rightarrow (8x+4y)dx + (4x+2y)dy = 0$$

$$\Phi(x,y;\lambda) = x^2 + xy + y^2 + \lambda(4x^2 + 4xy + y^2 - 1)$$

$$\Phi_x = 2x + y + 8\lambda x + 4y\lambda = 0$$

$$2x(1+4\lambda) + y(1+4\lambda) = 0$$

$$\frac{(1+4\lambda)(2x+y)}{(1+4\lambda)} = 0$$

$$\Phi_y = x + 2y + 4\lambda x + 2\lambda y = 0$$

$$x(1+4\lambda) + 2y(1+\lambda) = 0$$

$$4x^2 + 4xy + y^2 - 1 = 0$$

$$1^\circ \quad \lambda = -\frac{1}{4}$$

$$x(1+4(-\frac{1}{4})) + 2y(1-\frac{1}{4}) = 0$$

$$-\frac{6}{4}y = 0 \Rightarrow y = 0$$

$$4x^2 + 4xy + y^2 - 1 = 0$$

$$4x^2 - 1 = 0$$

$$(2x-1)(2x+1) = 0$$

$$x = \frac{1}{2} \vee x = -\frac{1}{2}$$

$$\left(\frac{1}{2}, 0\right) \quad \left(-\frac{1}{2}, 0\right)$$

$$\Phi_{xx} = 2 + 8\lambda$$

$$\Phi_{xy} = 1 + 4\lambda$$

$$\Phi_{yy} = 2 + 2\lambda$$

$$\Phi_{yx} = 1 + 4\lambda$$

$$d^2\Phi = (2+8\lambda)(dx)^2 + 2(1+4\lambda)dxdy + (2+2\lambda)(dy)^2$$

$$= (2+8(-\frac{1}{4}))(dx)^2 + 2(1+4(-\frac{1}{4}))dxdy + (2+2(-\frac{1}{4}))(dy)^2$$

$$= (2-2)(dx)^2 + 2(1-1)dxdy + (2-\frac{1}{2})(dy)^2$$

$$= \frac{3}{2}(dy)^2 > 0, \quad dy \neq 0$$

$\rightarrow$ f-ja ima minimum

$$2^\circ \quad y = -2x$$

$$x(1+4\lambda) - 4x(1+\lambda) = 0$$

$$x(1+4\lambda - 4 - 4\lambda) = 0$$

$$x = 0$$

$$y = -2x$$

$$y = 0$$

$$4x^2 + 4xy + y^2 - 1 = 0$$

$$y^2 - 1 = 0$$

$$(y-1)(y+1) = 0$$

$$y = 1 \vee y = -1$$

$$\left(\frac{1}{2}, 1\right) \quad \left(-\frac{1}{2}, -1\right)$$

$$b) f(x, y) = 2x^2 + 4xy - y^2, \text{ uz } \text{okol} \quad x^2 + y^2 = 1$$

$$\Phi(x, y, \lambda) = 2x^2 + 4xy - y^2 + \lambda(x^2 + y^2 - 1)$$

$$\Phi_x(x, y, \lambda) = 4x + 4y + 2\lambda x = 0 \quad 4y = -4x - 2\lambda x \quad 2y = -2x - \lambda x$$

$$\Phi_y(x, y, \lambda) = 4x - 2y + 2\lambda y = 0 \quad 4x = 2y - 2\lambda y \quad 2x = y - \lambda y$$

$$x = \frac{y(1-\lambda)}{2}$$

$$g_1(x, y) = x^2 + y^2 - 1 = 0$$

$$2y = -2 \frac{y(1-\lambda)}{2} - \frac{\lambda}{2} y(1-\lambda)$$

$$\frac{1}{4} y^2 (1-\lambda)^2 + y^2 - 1 = 0 \quad / \cdot 4$$

$$4y = -2y(1-\lambda) - \lambda y(1-\lambda)$$

$$y^2(1-2\lambda+\lambda^2) + 4y^2 - 4 = 0$$

$$4y = -2y + 2\lambda y - \lambda y + \lambda^2 y \quad (\Rightarrow) y(\lambda^2 + \lambda - 6) = 0$$

$$5y^2 - 2\lambda y^2 + y^2 \lambda^2 - 4 = 0$$

$$\begin{matrix} \uparrow \downarrow \\ y=0 \vee \lambda^2 + \lambda - 6 = 0 \end{matrix}$$

$$4 = -2 + \lambda + \lambda^2$$

$$y^2(5-2\lambda+\lambda^2) - 4 = 0$$

1) ako je $y=0$
kad ubacimo u drugu
jednačinu dobijamo
 $0 = -4$

$$2) \lambda^2 + \lambda - 6 = 0$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{25}}{2} = \frac{-1 \pm 5}{2}$$

$$\lambda_1 = -3 \quad \lambda_2 = 2$$

$$y^2 = \frac{4}{5-2\lambda+\lambda^2}$$

$$\lambda_1 = -3 \quad y^2 = \frac{4}{5+6+9} = \frac{4}{20} = \frac{1}{5} \quad y = \pm \frac{1}{\sqrt{5}}$$

$$\lambda_2 = 2 \quad y^2 = \frac{4}{5-4+4} = \frac{4}{5} \quad y = \pm \frac{2}{\sqrt{5}}$$

$$1^\circ \lambda = -3$$

$$y = \pm \frac{1}{\sqrt{5}}$$

$$x = \frac{1}{2} y(1-\lambda) = \frac{1}{2} \cdot \frac{1}{\sqrt{5}} \cdot 4 = \frac{2}{\sqrt{5}}$$

$$x = -\frac{2}{\sqrt{5}}$$

$$\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, -3 \right), \quad \left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, -3 \right)$$

$$2^\circ \lambda = 2$$

$$y = \pm \frac{2}{\sqrt{5}}$$

$$x = \frac{1}{2} y(1-\lambda) = \frac{1}{2} \cdot \frac{2}{\sqrt{5}} \cdot (-1) = -\frac{1}{\sqrt{5}}$$

$$x = \frac{1}{\sqrt{5}}$$

$$\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 2 \right), \quad \left(-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 2 \right)$$

b)

$$g_1(x, y) = x^2 + y^2 - 1 = 0$$

$$2x dx + 2y dy = 0$$

$$x dx = -y dy$$

$$\phi_{xx} = 4 + 2\lambda$$

$$\phi_{xy} = 4$$

$$\phi_{yy} = -2 + 2\lambda$$

$$\phi_{yx} = 4$$

$$d^2\phi = (4 + 2\lambda)(dx)^2 + 8 dx dy + (-2 + 2\lambda)(dy)^2$$

$$\lambda = -3$$

$$d^2\phi = -2(dx)^2 + 8 dx dy - 8(dy)^2$$

$$= -2(dx)^2 - 4 dx dy + 4(dy)^2$$

$$= -2(dx - 2dy)^2 < 0$$

f -ja ima maksimum

$$dx = -\frac{y}{x} dy$$

$$\lambda = 2$$

$$d^2\phi = 8(dx)^2 + 8 dx dy + 2(dy)^2$$

$$= 2(4(dx)^2 + 4 dx dy + (dy)^2)$$

$$= 2(2dx + dy)^2 > 0 \Rightarrow f\text{-ja ima minimum}$$

$$dx^2 \left(1 - 4 \frac{dy}{dx} + 4 \left(\frac{dy}{dx} \right)^2 \right)$$

$$dx^2 \left(1 + 4 \left(\frac{x}{y} \right) + 4 \left(\frac{dy}{dx} \right)^2 \right)$$

V
0

$$\lambda = -3 \quad \frac{x}{y} = 2 > 0$$

$$dx \neq 0$$

jer mora

biti jedan ili $\neq 0$

$$\text{a ako } dx=0 \Rightarrow dy=0$$

$$\text{jer } y/x \neq 0$$

19. ZADATAK

c) $f(x, y, z) = xy + yz$, uz uslov $x+y=2$ i $y+z=2$

$$\Phi(x, y, z; \lambda, \eta) = xy + yz + \lambda(x+y-2) + \eta(y+z-2)$$

$$\Phi_x = y + \lambda = 0 \Rightarrow y = -\lambda$$

$$\Phi_y = x + z + \lambda + \eta = 0$$

$$\Phi_z = y + \eta = 0 \Rightarrow y = -\eta$$

$$g_1 = x + y - 2 = 0$$

$$g_2 = y + z - 2 = 0$$

$$x + z = -\lambda - \eta$$

$$x + z = 2y$$

$$x + y = 2$$

$$y + z = 2$$

$$\rightarrow x = 2 - y$$

$$\rightarrow z = 2 - y$$

$$\Phi_{xx} = 0 \quad \Phi_{xy} = 1 \quad \Phi_{xz} = 0$$

$$\Phi_{yx} = 1 \quad \Phi_{yy} = 0 \quad \Phi_{yz} = 1$$

$$\Phi_{zx} = 0 \quad \Phi_{zy} = 1 \quad \Phi_{zz} = 0$$

$$2 - y + 2 - y = 2y$$

$$4 = 4y$$

$$y = 1$$

$$x = 2 - y = 1$$

$$z = 2 - y = 1$$

$$y = -\lambda \Rightarrow \lambda = -1$$

$$y = -\eta \Rightarrow \eta = -1$$

$$d^2\Phi = dx dy + dy dx + dy dz + dz dy$$

$$= 2(dx dy + dy dz) = 2 dy (dx + dz)$$

$$g_1 = x + y - 2$$

$$dx + dy = 0$$

$$dx = -dy$$

$$g_2 = y + z - 2$$

$$dy + dz = 0$$

$$dz = -dy$$

$$dy \neq 0$$

$$d^2\Phi = 2 dy (-2 dy) = -4(dy)^2 < 0$$

$$\Rightarrow f \text{ ima maksimum}$$

$$\sim (1, 1, 1)$$

$$f(1, 1, 1) = 1 + 1 = 2$$

stac. tačka za f je

$$(1, 1, 1)$$

stac. tačka za Φ je

$$(1, 1, 1, -1, -1)$$

20) Neka je $f(x,y) = x^2 + y^2$

a) Odrediti gradijent funkcije f

b) Odrediti gradijent f u tački $(3,2)$

c) Dati geometrijsku interpretaciju rezultata pod b)

a) Vektor (f_x, f_y) naziva se gradijent funkcije f

$$\nabla f = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) f = (f_x, f_y)$$

$$\nabla f = (2x, 2y)$$

$$b) \nabla f(3,2) = (2 \cdot 3, 2 \cdot 2) = (6, 4)$$

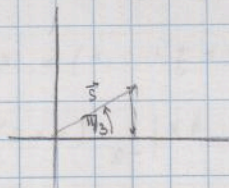
c) pravac u kome funkcija najbrže raste

21) Odkrediti izvod f -je $f(x,y) = x^2 - y^2$ po pravcu $\vec{s}$ koji zaklapa ugao od $\frac{\pi}{3}$ sa x -osom.

$$\vec{s} = \left(\cos \frac{\pi}{3}, \sin \frac{\pi}{3} \right) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$$

$$\nabla f = (2x, -2y)$$

$$\begin{aligned} \partial_{\vec{s}} f(x,y) &= \nabla f \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) = \frac{1}{2} f_x + \frac{\sqrt{3}}{2} f_y = \frac{1}{2} \cdot 2x + \frac{\sqrt{3}}{2} \cdot (-2y) \\ &= x - \sqrt{3}y \end{aligned}$$



22) Napisati jednačinu tangentne ravni na grafik funkcije $z = 2x^2 + 3y^2$ u tački $(-2, 1, z_0)$

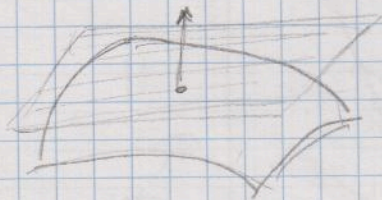
I način

$$z_0 = (-2, 1) = 2(-2)^2 + 3 \cdot 1^2 = 11$$

$$F(x,y,z) = 0$$

$$z = 2x^2 + 3y^2 \rightarrow 2x^2 + 3y^2 - z = 0$$

$$\vec{n} = \nabla F \Big|_{(-2,1)} = (4x, 6y, -1) \Big|_{(-2,1)} = (-8, 6, -1) \quad \text{- vektor normale}$$



jednačina ravni je $-8x + 6y - z = D$

$$-8x + 6y - z = 11$$

$$D = -8(-2) + 6 \cdot 1 - 11 = 16 + 6 - 11 = 11$$

→ Neka je data površ S jednačinom $z = f(x,y)$ i $(x_0, y_0, z_0) \in S$

Jednačina tangentne ravni u (x_0, y_0) je $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

→ Ako je površ data jednačinom $F(x,y,z) = 0$ onda je jednačina tangentne ravni na S u (x_0, y_0, z_0)

$$F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$$

nisu transformisani u koordinatu početka

II mačnu

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z = f(x, y) = 2x^2 + 3y^2 \quad (-2, 1, z_0) \in \text{površni}$$

$$z_0 = f(-2, 1) = 11$$

$$z - 11 = 4x \Big|_{-2} (x + 2) + 6y \Big|_1 (y - 1)$$

$$z - 11 = -8(x + 2) + 6(y - 1)$$

$$z - 11 = -8x - 16 + 6y - 6$$

$$8x - 6y + z = -11$$

(23) Odrediti vektor normale i jednačinu tangentne ravni na sferu

$$x^2 + y^2 + z^2 = 25 \text{ u tački } (1, 4, z_0), z_0 > 0.$$

- Gradjent $F(x, y, z) = x^2 + y^2 + z^2 - 25$ jednak je $\nabla F(x, y, z) = (2x, 2y, 2z)$

$$\nabla F(1, 4, z_0) = (2, 8, 4\sqrt{2}) \quad \text{— ovo je vektor normale tangentne ravni date površi u } (1, 4, 2\sqrt{2})$$

$$z^2 = 25 - x^2 - y^2 \quad z_0 > 0$$

$$z = \sqrt{25 - x^2 - y^2} \quad \swarrow \text{zato smo uzeli pozitivan koren}$$

$$z \Big|_{(1, 4)} = \sqrt{25 - 1 - 16} = \sqrt{8} = 2\sqrt{2} = z_0$$

- Jednačina tangentne ravni je :

$$\nabla f(1, 4, z_0) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

$$2(x - 1) + 8(y - 4) + 4\sqrt{2}(z - 2\sqrt{2}) = 0$$

$$2x - 2 + 4y - 32 + 4\sqrt{2}z - 16 = 0$$

$$2x + 4y + 4\sqrt{2}z = 50$$

24) Izračunati divergenciju i rotor vektorskog polja

$$\vec{F} = (2x - y^2, x^2 + 3z, 4y - z^2)$$

u tački $(1, 2, 3)$.

* DIVERGENCIJA vektorskog polja $\vec{F}$ je skalarno polje

$\nabla \cdot \vec{F} : D \rightarrow \mathbb{R}$ definisano sa :

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) (P, Q, R)$$

$$\vec{F}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z)) : D \rightarrow V$$

* ROTOR vektorskog polja $\vec{F}$, u oznaci $\nabla \times \vec{F}$ ili $\text{rot } \vec{F}$,

je vektorsko polje :

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{i} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \vec{j} \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \vec{k} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$\vec{F} = (2x - y^2, x^2 + 3z, 4y - z^2) \quad P = 2x - y^2, \quad Q = x^2 + 3z, \quad R = 4y - z^2$$

- divergencija

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = (2 + 0 - 2z) \Big|_{(1,2,3)} = 2 - 2z \Big|_{(1,2,3)} = -4$$

- rotor

$$\begin{aligned} \nabla \times \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y^2 & x^2 + 3z & 4y - z^2 \end{vmatrix} \Big|_{(1,2,3)} = \vec{i} \cdot 4 + \vec{j} \cdot 0 + \vec{k} \cdot 2x \\ &\quad - \vec{j} \cdot 0 - \vec{i} \cdot 3 - \vec{k} \cdot (-2y) \\ &= \vec{i} (4 - 3) + \vec{k} (2x + 2y) \Big|_{(1,2,3)} = \vec{i} + 6\vec{k} \end{aligned}$$