Absolutely Monotone Real Set Functions

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***Abstract*—We present a class of absolutely monotone and signed stable set functions with $m(\varnothing) = 0$, $AMSS$. The representation of a set function from $AMSS$ as a symmetric maximum of two monotone set function is obtained. We present three integrals of a real-valued measurable function based on $m \in AMSS$.**

Keywords: symmetric maximum, absolutely monotone set function, sign stable set function

I. Introduction

A generalization of the classical probability measure, a fuzzy measure (capacity), together with fuzzy integrals, has many applications in economics, pattern recognition, and decision analysis [3], [4], [5], [20], [21]. It is proven in [1], [16] that a real-valued set function m, $m(\varnothing)$, belongs to the space of set functions with the bounded chain variation, BV, if it can be represented as a difference of two finite fuzzy measures, vanishing at the empty set, i.e. $m = \nu_1 - \nu_2$.

The most important integral defined with respect to a fuzzy measure is introduced by G. Choquet in [2]. The Choquet integral of a non-negative real-valued function f based on a fuzzy measure $m : \mathcal{A} \to [0, \infty]$ is defined by

$$C_m(f) = \int_0^\infty m(\{x | f(x) \geq t\})dt. \tag{1}$$

The Choquet integral is defined with respect to non-monotonic, real-valued set functions with the bounded chain variation [14], and it is also known as the asymmetric Choquet integral [3], [14], [16]. The asymmetric Choquet integral is linear with respect to m.

An another important integral based on a fuzzy measure $m : \mathcal{A} \to [0, 1]$, introduced in [18] by M. Sugeno, is the Sugeno integral defined by

$$S_m(f) = \sup_{t \in [0,1[} (t \wedge m(\{x | f(x) \geq t\})). \tag{2}$$

The symmetric Sugeno integral of a real-valued function, introduced in [5], is also defined with respect to a fuzzy measure.

The Choquet-like integral related to some non-decreasing function $g : [0, 1] \to [0, \infty]$, $g(0) = 0$, defined for a non-negative function and a fuzzy measure m is given by

$$C_m^g(f) = g^{-1}\left(C_{g \circ m}(g \circ f)\right). \tag{3}$$

This integral is introduced in [9] and it is also defined for a real-valued function f, if for g is taken its odd extension on the real line [9].

In [13] the authors introduced an absolutely monotone and sign stabile set function $m : \mathcal{A} \to [-1, 1]$, $m(\varnothing) = 0$. It is shown that m can be represented by a pseudo-difference of two fuzzy measures. The class of such set functions is denoted by $AMSS$.

The aim of this paper is to present the class $AMSS$, and to propose different types of monotonic integrals based on a set function from $AMSS$. Our previous papers [12], [13] are devoted to non-monotonic integrals.

The paper is organized as follows. In Section 2 the preliminary notions and definitions are given. In Section 3 we consider the class of absolutely monotone and sign stabile set functions, $AMSS$. In Section 4 we propose definitions of three integrals based on $m \in AMSS$, $m(X) < 1$ and present their basic properties.

II. Preliminaries

The symmetric maximum is originally introduced in [5], [6].

Definition 1: The symmetric maximum $\otimes : [-a, a]^2 \to [-a, a]$, $a \in \overline{\mathbb{R}}^+$ is given by

$$x \otimes y \quad := \quad \operatorname{sign}(x + y)(|x| \vee |y|).$$

The symmetric maximum is a commutative, non-decreasing operation with neutral element 0 and anihilator a. It is not associative, nor continious.

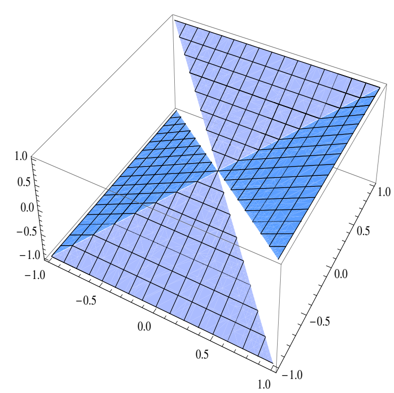

Figure 1. Symmetric maximum

Let I be a set of elements from $[-a,a]$. The symmetric maximum is defined by:

$$\varovee_{i\in I} x_i = \left(\varovee_{x_i\geqslant 0} x_i\right) \varovee \left(\varovee_{x_i<0} x_i\right) = \sup_{x_i\geqslant 0} x_i \varovee \inf_{x_i<0} x_i.$$

Let us assume X is a non-empty universal set. Let $\mathcal{A}$ be a σ-algebra of subsets of X.

Definition 2: A set function, $m : \mathcal{A} \to [-a,a]$, $a \in \overline{\mathbb{R}}^+$, $m(\varnothing) = 0$ is

(i) *non-negative* if $m(A) \geqslant 0$ for all $A \in \mathcal{A}$,

(ii) *revised monotone* if for all $A, B \in \mathcal{A}$, $A \cap B = \varnothing$, we have

a) $m(A) \geqslant 0$, $m(B) \geqslant 0$, $m(A) \vee m(B) > 0 \Rightarrow$

$$m(A \cup B) \geqslant m(A) \vee m(B);$$

b) $m(A) \leqslant 0$, $m(B) \leqslant 0$, $m(A) \wedge m(B) < 0 \Rightarrow$

$$m(A \cup B) \leqslant m(A) \wedge m(B);$$

c) $m(A) > 0$, $m(B) < 0 \Rightarrow$

$$m(B) \leqslant m(A \cup B) \leqslant m(A).$$

(iii) *fuzzy measure* if it is non-negative and monotone, i.e. if for all A, $B \in \mathcal{A}$, $A \subset B$ we have $0 \leqslant m(A) \leqslant m(B)$,

(iv) *signed fuzzy measure* if it is revised monotone.

We introduce the notion of absolutely monotone set functions.

Definition 3: A set function $m : \mathcal{A} \to \overline{\mathbb{R}}$ is

(i) absolutely monotone if for all $A, B \in \mathcal{A}$, $A \subset B$ we have $|m(A)| \leqslant |m(B)|$.

(ii) absolutely strictly monotone if for all $A, B \in \mathcal{A}$, $A \subset B$, $A \neq B$, we have $|m(A)| < |m(B)|$.

Obviously, any fuzzy measure is an absolutely monotone set function.

Example 1: Let $X = A \cup B$, $A \cap B = \varnothing$, such that $card(A) = card(B) = n$. Let m be defined on $\mathcal{P}(X)$ by

$$m(E) = \begin{cases} card(E), & E \subset A, \\ -card(E) + 1, & E \subset B, E \neq \varnothing \\ -n, & else. \end{cases}$$

The set function m is absolutely monotone, but it is not strictly absolutely monotone.

A signed fuzzy measure is not an absolutely monotone set function in general, as it is illustrated in the next example from [13].

Example 2: Let $X = \{1,2\}$ and let m be a signed fuzzy measure defined on $\mathcal{P}(X)$ by: $m(\varnothing) = 0$, $m(\{1\}) = 3$, $m(\{2\}) = -2$, $m(\{1,2\}) = 1$. We have $m(\{1,2\}) < m(\{1\})$, hence m is not absolutely monotone.

For a set function m we define its positive and negative part $m^+, m^- : \mathcal{A} \to [0,a]$ by:

$$m^+(A) = m(A) \vee 0, \tag{4}$$
$$m^-(A) = (-m(A)) \vee 0, \tag{5}$$

Maxitive fuzzy measures are considered in [16], [22].

Definition 4: A fuzzy measure m is *completely maxitive* if for any family $\{A_i\}_{i\in I}$ from $\mathcal{A}$ such that $\bigcup_{i\in I} A_i \in \mathcal{A}$ we have

$$m(\bigcup_{i\in I} A_i) = \sup_{i\in I} m(A_i).$$

$\varovee$-measures are introduced in [11].

Definition 5: A set function, $m : \mathcal{A} \to [-a,a]$, is a $\varovee$-*measure* if it satisfies

$$m(A \cup B) = m(A) \varovee m(B)$$

for all A, $B \in \mathcal{A}$, $A \cap B = \varnothing$.

Note that each $\varovee$-measure is revised monotone (Definition 2 (ii)), hence it is a signed fuzzy measure.

Example 3: Let $X = \{x_1, x_2, \ldots, x_n\}$. Let m be a set function $m : \mathcal{P}(X) \longrightarrow [-1,1]$ with $m(\varnothing) = 0$, defined by:

$$m(A) = \begin{cases} \frac{1}{\min\limits_{x_i\in A} i} & if \ \min\limits_{x_i\in A} i = 2k \\ -\frac{1}{\min\limits_{x_i\in A} i} & if \ \min\limits_{x_i\in A} i = 2k+1 \end{cases}$$

m is a $\varovee$-measure.

III. Space of real set functions $AMSS$

In this section we present results proven in [13] concerning to the space $AMSS$ and the representation of a real-valued set function from $AMSS$ by a symmetric maximum of two monotone set functions.

Definition 6: Let $m : \mathcal{A} \to [-a,a]$, $a \in \overline{\mathbb{R}}^+$, $m(\varnothing) = 0$ be a set function.

(i) We say m is a sign stable if it fulfils:

$$\begin{aligned} \sup_{E\subset A} m^+(E) &< m^-(A), \quad if\ m(A) < 0, \\ \sup_{E\subset A} m^-(E) &< m^+(A), \quad if\ m(A) > 0, \\ \text{for all } E \subset A \quad m^+(E) &= m^-(E), \quad if\ m(A) = 0 \end{aligned}$$

(ii) If m is sign stabile and absolutely monotone, we say m belongs to the class $AMSS$.

Lemma 1: Let $m_1, m_2 : \mathcal{A} \to [0,a]$ be fuzzy measures such that for each $A \subset B$, $A, B \in \mathcal{A}$ we have $m_1(A) = m_2(A)$, whenever $m_1(B) = m_2(B)$. Then a set function $m : \mathcal{A} \to [-a,a]$ defined by $m(A) := m_1(A) \varovee (-m_2(A))$, for $A \in \mathcal{A}$ is absolutely monotone.

Proposition 1: Each absolutely strictly monotone set function defined on a finite $\mathcal{A}$, belongs to $AMSS$.

We have the next representation theorem of a set function from $AMSS$.

Theorem 1: Let $m : \mathcal{A} \to [-a,a]$, $a \in \overline{\mathbb{R}}^+$ be from $AMSS$, such that $m(\varnothing) = 0$. Then, there exist two fuzzy measures m_1, $m_2 : \mathcal{A} \to [0,a]$ such that $m = m_1 \varovee (-m_2)$ and $|m| = m_1 \vee m_2$, where $m_1 : \mathcal{A} \to [0,a]$ is defined by:

$$m_1(A) = \begin{cases} m^+(A), & m(A) \geqslant 0, \\ \sup\limits_{E\subset A} m^+(E), & m(A) < 0 \end{cases} \tag{6}$$

and $m_2 : \mathcal{A} \to [0,a]$ is defined by:

$$m_2(A) = \begin{cases} m^-(A), & m(A) \leqslant 0, \\ \sup_{E \subset A} m^-(E), & m(A) > 0. \end{cases} \quad (7)$$

Example 4: Let m be a set function defined on the subclass of Borel subsets of the interval $[-a,a]$. For each $[c,d] \in \mathcal{B}([-a,a])$, $-a \leqslant c \leqslant d \leqslant a$, $a \in \overline{\mathbb{R}}^+$ m is defined by:

$$m([c,d]) = \begin{cases} d-c, & d > 0, \\ c-d, & d \leqslant 0. \end{cases}$$

m is an absolutely monotone set function. If $a < \infty$, then m is a sign stable on class of Borel subsets having form $[c,d]$. Hence, there exist m_1 and m_2 defined with (6), (7), given by:

$$m_1([c,d]) = \begin{cases} d-c, & d > 0, \\ 0, & d \leqslant 0, \end{cases}$$

$$m_2([c,d]) = \begin{cases} 0, & d > 0, c > 0 \\ -c, & d > 0, c \leqslant 0 \\ d-c, & d \leqslant 0. \end{cases}$$

Obviously, m_1 and m_2 are fuzzy measures and they induce measures m_1 and m_2 on $\mathcal{B}([-a,a])$. Together with Theorem 1 this fact implies

$$m(B) = m_1(B) \circledvee (-m_2(B)), \quad \text{for each } B \in \mathcal{B}([-a,a]).$$

Theorem 2: Let $m : \mathcal{A} \to [-a,a]$, $a \in \overline{\mathbb{R}}^+$, such that $m(\varnothing) = 0$.
(i) $m \in AMSS$ if and only if there exist fuzzy measures m_1, $m_2 : \mathcal{A} \to [0,a]$, such that for each $A \subset B$, $A, B \in \mathcal{A}$ we have $m_1(A) = m_2(A)$ whenever $m_1(B) = m_2(B)$, and

$$m = m_1 \circledvee (-m_2).$$

(ii) Moreover, if $m \in AMSS$ and $m = \tilde{m}_1 \circledvee (-\tilde{m}_2)$ for some fuzzy measures $\tilde{m}_1$, $\tilde{m}_2$, then $m_1 \leqslant \tilde{m}_1$ and $m_2 \leqslant \tilde{m}_2$, where m_1 and m_2 are defined by (6) and (7).

Example 5: Let $X = \{1,2,3,4\}$ and m be a $\circledvee$-measure such that $m(\varnothing) = 0$, defined on $\mathcal{P}(X)$ by:

$m(\{1\})=0$	$m(\{2\})=-3$	$m(\{3\})=-4$
$m(\{4\})=5$	$m(\{1,2\})=-3$	$m(\{1,3\})=-4$
$m(\{1,4\})=5$	$m(\{2,3\})=-4$	$m(\{2,4\})=5$
$m(\{3,4\})=5$	$m(\{1,2,3\})=-4$	$m(\{1,2,4\})=5$
$m(\{2,3,4\})=5$	$m(\{1,3,4\})=5$	$m(X)=5$

$m \in AMSS$ and the lowest fuzzy measures (in the sense of Theorem 2 (ii)) such that $m = m_1 \circledvee (-m_2)$ are m_1 and m_2 given by (6) and (7):

$$m_1(A) = m^+(A) \quad \text{for all} \quad A \in \mathcal{P}(X) \quad \text{and}$$

$m_2(\{1\})=0$	$m_2(\{2\})=3$	$m_2(\{3\})=4$
$m_2(\{4\})=0$	$m_2(\{1,2\})=3$	$m_2(\{1,3\})=4$
$m_2(\{1,4\})=0$	$m_2(\{2,3\})=4$	$m_2(\{2,4\})=3$
$m_2(\{3,4\})=4$	$m_2(\{1,2,3\})=4$	$m_2(\{1,2,4\})=3$
$m_2(\{2,3,4\})=4$	$m_2(\{1,3,4\})=4$	$m(X)=4$

The relationship of $m \in AMSS$ and a $\circledvee$-measure is established in the next theorems. In the sequel $\mathcal{A} = \mathcal{P}(X)$.

Theorem 3: Let $m \in AMSS$, $m(\varnothing) = 0$. If m_1 and m_2 given by Theorem 1 are completely maxitive fuzzy measures defined by (6) and (7), respectively, then m is a $\circledvee$-measure.

Theorem 4: Let m be a $\circledvee$-measure such that each subset of m-null-set is m-null-set, i.e. for each $A \subset B$, $A, B \in \mathcal{A}$ we have $m(A) = 0$, whenever $m(B) = 0$. Then $m \in AMSS$.

IV. Integrals based on $m \in AMSS$

Let $(X, \mathcal{A})$ be a measurable space, where X is a universal set. Let $f : X \to [-1,1]$ be an $\mathcal{A}$-measurable, such that $\sup_{x \in X} f(x) < 1$. We denote with $\mathcal{F}$ the class of such functions. We propose three different types of integrals of function $f \in \mathcal{F}$. The first one is related to the Sugeno integral, the second is related to the Choquet-like integral, and the third one can be obtained as a limit of the sequence of integrals of second type.

Let $g : [-1,1] \to [-\infty,\infty]$, $g(0) = 0$ be an odd, strictly increasing, continuous function. A pseudo-addition $\oplus : [-1,1]^2 \to [-1,1]$ is defined by

$$x \oplus y = g^{-1}(g(x) + g(y)); \quad (8)$$

with the convention $\infty - \infty = \infty$ or $\infty - \infty = -\infty$.

A pseudo-addition $\oplus$ is associative, commutative, strictly monotone on $]-1,1[^2$, continuous up to $(1,1)$,$(-1,-1)$, with neutral element 0. It is a symmetric operation, i.e. for all $x, y \in [-1,1]^2 \setminus \{(1,1),(-1,-1)\}$ we have

$$-(x \oplus y) = (-x) \oplus (-y).$$

The pseudo-difference $\ominus$ with respect to $\oplus$ for $x, y \in [-1,1]^2 \setminus \{(1,1),(-1,-1)\}$ is given by

$$x \ominus y = x \oplus (-y).$$

A pseudo-multiplication $\odot : [-1,1]^2 \to [-1,1]$, distributive with respect to $\oplus$, can be defined by the additive generator of pseudo-addition $\oplus$, see [6], [7], [10], [12]. The pseudo-multiplication $\odot : [-1,1]^2 \to [-1,1]$ is defined by:

$$x \odot y = g^{-1}(g(x)g(y)), \quad (9)$$

with the convention $\infty \cdot 0 = 0$ or $\infty \cdot 0 = \infty$.

Definition 7: Let $m \in AMSS$, $|m(X)| < 1$ and $f \in \mathcal{F}$. We define

(1) *Asymmetric Sugeno integral of f based on m*:

$$S_m(f) = S_{m_1}(f^+) \circledvee (-S_{m_2}(f^-)),$$

(2) *Generated Choquet integral of f based on m*:

$$C_m^g(f) = C_{m_1}^g(f^+) \ominus C_{m_2}^g(f^-),$$

(3) *Asymmetric $(\bigcirc\!\!\!\!\vee, \odot)$-integral of f based on m*:

$$ASI_m(f) \quad = \quad \sup_{t\in[0,1[} (t \odot m_1(\{f^+ \geqslant t\})) \bigcirc\!\!\!\!\vee$$
$$\bigcirc\!\!\!\!\vee \Big(- \sup_{t\in[0,1[} (t \odot m_2(\{f^- \geqslant t\}))\Big),$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

Proposition 2: Let $m \in AMSS$, such that $|m(X)| < 1$ and let $g : [-1, 1] \to [-\infty, \infty]$, $g(0) = 0$ be an odd, strictly increasing, continuous function. Then there exists a sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ such that for all $f \in \mathcal{F}$ we have

$$\lim_{n\to\infty} C_m^{g_n}(f) = ASI_m(f).$$

Proposition 3: The integrals proposed in Definition 7 are monotone and for all $m \in AMSS$, $|m(X)| < 1$ and $f \in \mathcal{F}$ we have

$$I_m(-f) = -I_{-m}(f),$$

where $-m \in AMSS$ and $| - m(X)| < 1$.

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