Decidability and Complexity of Some Interpretability Logics

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Basic interpretability logic **IL**

- Intepretability logics have a binary modal operator $\rhd$.
- Basic interpretability logic **IL**:

  classically valid formulas (in the new language with $\Box, \Diamond, \rhd$);

  K $\Box(A \to B) \to (\Box A \to \Box B)$;

  Löb $\Box(\Box A \to A) \to \Box A$;

  J1 $\Box(A \to B) \to A \rhd B$;

  J2 $(A \rhd B) \ \wedge \ (B \rhd C) \to A \rhd C$;

  J3 $(A \rhd C) \ \wedge \ (B \rhd C) \to A \vee B \rhd C$;

  J4 $A \rhd B \to (\Diamond A \to \Diamond B)$;

  J5 $\Diamond A \rhd A$.

  - rules: modus ponens and necessitation $A / \Box A$.

  (parentheses priority: $\neg, \Box, \Diamond; \ \wedge, \vee; \ \rhd; \ \to, \leftrightarrow$)

Models

- Semantics: extend the usual relational (Kripke) model.
- **IL**-frame (Veltman frame): $\mathcal{F} = \langle W, R, \{S_w : w \in W\}\rangle$, where:
  1. $W \neq \emptyset$;
  2. R^{-1} is well-founded (no $x_0 R x_1 R x_2 R \ldots$ chains);
  3. R is transitive;
  4. $S_w \subseteq R(w)^2$ is reflexive, transitive, contains $R \cap R(w)^2$ ($wRuRv$ implies $uS_w v$);
- **IL**-model (Veltman model): $\mathcal{M} = \langle W, R, \underline{\{S_w : w \in W\}}, V\rangle$, where:
  1. $\langle W, R, \{S_w : w \in W\}\rangle$ is a **IL**-frame;
  2. $V \subseteq W \times Prop$ (or $V : Prop \to \mathcal{P}(W)$).

Models (2)

- Veltman model: $\mathcal{M} = \langle W, R, \underline{\{S_w \ : \ w \in W\}}, V \rangle$.
- $w \Vdash p$ if and only if wVp, for $p \in Prop$.
- Logical connectives have classical semantics.
- Truth of a formula $F \rhd G$ ("F interprets G") in a world $w \in \mathcal{M}$:

$$w \Vdash F \rhd G \quad :\Leftrightarrow \quad \forall x \in R(w) : \quad x \Vdash F \Rightarrow \exists y \in S_w(x) : \ y \Vdash G.$$

- Soundness and completenss:

$$\mathbf{IL} \vdash F \quad \Leftrightarrow \quad \forall \mathcal{F} : \ \mathcal{F} \vDash F.$$

Extensions and frame conditions

- Some extensions of **IL**:

$\mathbf{IL}M_0$	$\mathbf{IL} + A \rhd B \to \Diamond A \land \Box C \rhd B \land \Box C$
$\mathbf{IL}W$	$\mathbf{IL} + A \rhd B \to A \rhd B \land \Box\neg A$
$\mathbf{IL}W^*$	$\mathbf{IL} + A \rhd B \to B \land \Box C \rhd B \land \Box C \land \Box\neg A$

- $\mathbf{IL}W^* = \mathbf{IL}M_0 W \subseteq \mathbf{IL}(All)$
- These logics are complete w.r.t. certain classes of frames:

(M_0)	$wRuRxS_w v \Rightarrow R(v) \subseteq R(u)$;
(W)	$S_w \circ R$ is reverse well-founded for each w;
(W^*)	(M_0) and (W).

- **IL**W-frame is **IL**-frame that satisfies (W) etc.

Proving decidability

- FMP: if $x \nVdash F$, then there is finite $\mathcal{M}$ and $x' \in \mathcal{M}$ s.t. $x' \nVdash F$.
- Decision procedure: simultaneously do two things:
  - Enumerate the (countable) set of all **IL**-proofs.
  - Enumerate the (countable) set of (descriptions of) finite **IL**-models.
- The usual way of proving FMP is by filtrations.

Filtrations on **IL**-models

- Let Γ contain A, closed under subformulas.
- Assume $\sim$ is an equivalence relation on W, $\sim \subseteq \equiv_\Gamma$.
- For any $V \subseteq W$, define $\widetilde{V} = \{[v] \mid v \in V\}$.
- We define the rest of $\widetilde{M}$ as follows.
- $\widetilde{R} = \{([w],[u]) \mid wRu, \exists \Box C \in \Gamma : w \nVdash \Box C, u \Vdash \Box C\}$.
- Define $\Vdash$ so that x and $[x]$ agree on variables in Γ.
- Problem: how to define $S_{[w]}$.
  "Generous" definitions do not preserve transitivity; while "strict" definitions lose $S_{[w]}$-witnesses for some $\rhd$-formulas.
- Solution: a more fine-grained semantics, where $S_{[w]}$-witnesses are not complete sets of formulas.

Problems with filtrations on **IL**-models

(1. try) $[u]\widetilde{S}_{[w]}[v]$ if and only if $[u],[v] \in \widetilde{R}([w])$, and for <u>all/some</u> $w' \in [w]$ and <u>some</u> $u' \in [u]$ such that $w'Ru'$ we have $u'S_{w'}v'$ for some $v' \sim v$.

- $w \to \{u \rightsquigarrow v_1 \sim v_2 \rightsquigarrow z\}$, $[w] \to \{[u] \rightsquigarrow [v] \rightsquigarrow [z]\}$
- But, we do not have $[u] \rightsquigarrow [z]$.
- If transitivity forced, some false $\rhd$-formula might get its witness and become true.

(2. try) $[u]\widetilde{S}_{[w]}[v]$ if and only if $[u],[v] \in \widetilde{R}([w])$, and for <u>all/some</u> $w' \in [w]$ and <u>all</u> $u' \in [u]$ such that $w'Ru'$ we have $u'S_{w'}v'$ for some $v' \sim v$.

- $w \to \{v_1[X] \leftsquigarrow u_1 \sim u_2 \rightsquigarrow v_2[\neg X]\}$,
  $[w] \to \{[u] \rightsquigarrow \quad ? \quad \}$
- Problem: we lose S_w-successors that do not agree enough.

Generalized models

- Generalized **IL**-models (generalized Veltman models).
- $\mathcal{M} = \langle W, R, \underline{\{S_w \ : \ w \in W\}}, V \rangle$, where:
  1. $W \neq \emptyset$;
  2. R^{-1} is well-founded (no $x_0 R x_1 R x_2 R \ldots$ chains);
  3. R is transitive;
  4. $S_w \subseteq R(w) \times (2^{R(w)} \setminus \{\emptyset\})$ is:
     - quasi-reflexive $uS_w\{u\}$;
     - quasi-transitive $uS_w\{v_i \mid i \in I\}$ and $v_i S_w Z_i \Rightarrow uS_w \bigcup\{Z_i \mid i \in I\}$;
     - contains $R \cap R(w)^2$ $wRuRv$ implies $uS_w\{v\}$;
     - is monotonous $uS_w V \Rightarrow uS_w V', V \subseteq V'$
  5. $V \subseteq W \times Prop$ (or $V : Prop \to \mathcal{P}(W)$).
- Truth of a formula $F \rhd G$ ("F interprets G") in a world $x \in \mathcal{M}$:

$$w \Vdash F \rhd G \quad :\Leftrightarrow \quad \forall x \in R(w) : \ \ x \Vdash F \Rightarrow \exists V \in S_w(x) : \ V \Vdash G.$$

- $V \Vdash G$ stands for $v \Vdash G$ for all $v \in V$.

Filtration property

- $\widetilde{\mathcal{M}} = \langle \widetilde{W}, \widetilde{R}, \widetilde{S}_{[w]}, \Vdash \rangle$.
- $\widetilde{W} = \{[w] \mid w \in W\}$.
- $\widetilde{R} = \{([w],[u]) \mid wRu, \exists \Box C \in \Gamma : w \nVdash \Box C, u \Vdash \Box C\}$.
- $[u]\widetilde{S}_{[w]}\widetilde{V}$ if and only if $\{[u]\}, \widetilde{V} \subseteq R([w])$, and for all $w' \in [w]$ and all $u' \in [u]$ such that $w'Ru'$ we have $u'S_{w'}V(w',u')$ for some $V(\widetilde{w',u'}) \subseteq \widetilde{V}$.
- Forcing relation compatible with $\mathcal{M}$.
- Assume $\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ is a generalized model (depends on $\sim$).
- Do we have $w \Vdash F \iff [w] \Vdash F$?

Theorem

$w \Vdash F \iff [w] \Vdash F.$

- So, if $\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ is a model at all, then it is a filtration of $\mathcal{M} = \langle W, R, S, \Vdash \rangle$.
- Is it a model (does it satisfy quasi-transitivity etc.)? Depends on what $\sim$ is.
- Ideally, x and $[x]$ are structurally similar, so that quasi-transitivity etc. is preserved.
- So, each $y \sim x$ should be structurally similar to x.

Definition

A *bisimulation* between generalized **IL**-models $\langle W, R, \{S_w : w \in W\}, \Vdash\rangle$ and $\langle W', R', \{S'_{w'} : w' \in W'\}, \Vdash\rangle$ is any $Z \subseteq W \times W'$, $Z \neq \emptyset$:

(at) if wZw' then $w \Vdash p \iff w' \Vdash p$;

(forth) if wZw' and wRu, then there exists $u' \in R'(w')$ with uZu' and for all $V' \in S'_{w'}(u')$ there is $V \in S_w(u)$ such that for all $v \in V$ there is $v' \in V'$ with vZv';

(back) if wZw' and $w'R'u'$, then there exists $u \in R(w)$ such that uZu' and for all $V \in S_w(u)$ there is $V' \in S'_{w'}(u')$ such that for all $v' \in V'$ there is $v \in V$ with vZv'.

- By induction on F, if x and y are bisimilar (w.r.t. any bisimulation), $x \Vdash F \iff y \Vdash F$.
- Union of bisimulations (over generalized models) is itself a bisimulation (*Vrgoč and Vuković, 2010*).
- In particular, there is a largest (auto)bisimulation $Z \subseteq W^2$.

- Denote by $\sim$ the largest bisimulation on W^2.

  (equivalently, denote $x \sim y$ if there is any bisimulation at all which equates x and y)

Theorem

$\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ is a model.

- Thus, if $\sim$ is the largest bisimulation on W^2, then $\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ is a model, and a filtration.

  We were trying to prove finite model property; is this a finite model?

- Each $\widetilde{R}$-transition eliminates at least one $\Diamond$-formula from Γ; so height is finite.
- Still, branching factor might be infinite.

Definition

A *n-bisimulation* between **IL**-models $\langle W, R, \{S_w : w \in W\}, \Vdash\rangle$ and $\langle W', R', \{S'_{w'} : w' \in W'\}, \Vdash\rangle$ is any sequence $Z_n \subseteq \cdots \subseteq Z_0 \subseteq W \times W'$:

(at) if wZ_0w' then $w \Vdash p \iff w' \Vdash p$;

(forth) if wZ_nw' and wRu, then there exists $u' \in R'(w')$ with $uZ_{n-1}u'$ and for all $V' \in S'_{w'}(u')$ there is $V \in S_w(u)$ such that for all $v \in V$ there is $v' \in V'$ with $vZ_{n-1}v'$;

(back) if wZ_nw' and $w'R'u'$, then there exists $u \in R(w)$ such that $uZ_{n-1}u'$ and for all $V \in S_w(u)$ there is $V' \in S'_{w'}(u')$ such that for all $v' \in V'$ there is $v \in V$ with $vZ_{n-1}v'$.

- Since height of $\mathcal{M}$ is bounded by $|\Gamma|$, worlds are $|\Gamma|$-bisimilar iff bisimilar.

- Put $u \equiv_n v$ if u and v agree on all formulas with at most n nested modalities.
- From now on, assume $Prop := Prop \cap \Gamma$.
- Now there are only finitely many formulas of modal depth up to $|\Gamma|$ (finitely many up to local equivalence).
- Denote $Th_n\, w$ the set of all formulas F with modal depth up to $|\Gamma|$ and $w \Vdash F$.

Lemma

$u \sim_n v \iff u \equiv_n v.$

- Denote $\mathcal{N} = \widetilde{\mathcal{M}}$.
- For $x, y \in \mathcal{N}$, we now have $x \sim y \iff x \sim_{|\Gamma|} y \iff x \equiv_\Gamma y$.
- There are obviously only finitely many worlds in $\mathcal{M}/\equiv_\Gamma$.
- Since $\equiv_\Gamma \; = \; \sim_{|\Gamma|}$, $\widetilde{\mathcal{N}}$ (that is, $\widetilde{\widetilde{\mathcal{M}}}$) has only finitely many worlds.
- Thus we have FMP for **IL**.

Extending to **IL**X

- To prove FMP, given **IL**X that is complete w.r.t. class of Veltman frames that satisfy property $\mathcal{C}$, we need to fill in the following:
    1. What is the (generalized) frame condition $\mathcal{G}$ of X?
    2. Is **IL**X complete w.r.t. to the class of $\mathcal{G}$-frames?
    3. Does $\widetilde{\mathcal{M}}$ have $\mathcal{G}$ if $\mathcal{M}$ has $\mathcal{G}$?
- For popular choices of X (except for W, W*), 1 is known; and 2 usually reduces to completeness w.r.t. $\mathcal{C}$ (for each VM take the natural GVM, i.e. $uS_w v \Rightarrow uS_w\{v\}$).

Logic $\mathbf{IL}\mathrm{M}_0$

- $\mathbf{IL}\mathrm{M}_0$ is $\mathbf{IL} + A \rhd B \to \Diamond A \wedge \Box C \rhd B \wedge \Box C$.
- Frame condition (M_0):

$$wRuRxS_w vRz \Rightarrow uRz.$$

- Frame condition $(M_0)_{gen}$:

$$wRuRxS_w V \Rightarrow (\exists V' \subseteq V)(uS_W V' \ \&\ R(V') \subseteq R(u)).$$

- For each VM with (M_0), there is a natural GVM (put $xS_W\{y\}$ whenever $xS_w y$) with $(M_0)_{gen}$.
- Remains to prove $\widetilde{\mathcal{M}}$ preserves $(M_0)_{gen}$.

Theorem

If $\mathcal{M}$ has property $(M_0)_{gen}$, then $\widetilde{\mathcal{M}}$ has property $(M_0)_{gen}$.

Logic $\mathbf{IL}\mathsf{W}$

- $\mathbf{IL}\mathsf{W}$ is $\mathbf{IL} + A \rhd B \to A \rhd B \wedge \Box\neg A$.
- Frame condition (W):

$$S_w \circ R \text{ is reverse well-founded for each } w$$

- Frame condition $(W)_{gen}$?

$$(\forall w \in W)(\forall X \subseteq R[w])(\forall z \in W)$$
$$\Big(zS_wX \Rightarrow (\exists V \subseteq X)\big(zS_wV \;\&\; (\forall v \in V)(R[v] \cap S_w^{-1}[X] = \emptyset)\big)\Big)$$

- For each VM with (W), there is a natural GVM (put $xS_W\{y\}$ whenever xS_wy) with $(W)_{gen}$.

Theorem

If $\mathcal{M}$ has property $(W)_{gen}$, then $\widetilde{\mathcal{M}}$ has property $(W)_{gen}$.

Logic $\mathbf{IL}\mathrm{W}^*$

- $\mathbf{IL}\mathrm{W}^*$ is $\mathbf{IL} + A \rhd B \to B \wedge \Box C \rhd B \wedge \Box C \wedge \Box\neg A$.
- $\mathbf{IL}\mathrm{W}^* = \mathbf{IL}\mathrm{WM}_0$.
- Frame condition $(W^*)_{gen}$?
- Each $\mathbf{IL}\mathrm{W}^*$-frame is $\mathbf{IL}\mathrm{W}$-frame ($\mathbf{IL}\mathrm{WM}_0 \supseteq \mathbf{IL}\mathrm{W}$) and $\mathbf{IL}\mathrm{M}_0$-frame ($\mathbf{IL}\mathrm{WM}_0 \supseteq \mathbf{IL}\mathrm{M}_0$).
- Conversely, if $\mathcal{F}$ is both an $\mathbf{IL}\mathrm{W}$-frame and an $\mathbf{IL}\mathrm{M}_0$-frame, then it is an $\mathbf{IL}\mathrm{WM}_0$-frame (induction on proof length).
- So, the frame condition is:

$$(W)_{gen} \text{ and } (M_0)_{gen}.$$

- If $\mathbf{IL}W^* \nvdash F$, there is a $\mathbf{IL}\mathrm{M}_0$-, $\mathbf{IL}\mathrm{W}$-VM $\mathcal{M}$, $w \in \mathcal{M}$, s.t. $w \nVdash F$. Then there is a natural GVM $\mathcal{N}$ with similar properties. Then $\widetilde{\mathcal{N}}$ is an $\mathbf{IL}\mathrm{M}_0$-, $\mathbf{IL}\mathrm{W}$-GVM, and so an $\mathbf{IL}\mathrm{W}^*$-GVM.

Complexity

- Given X, what is comp. complexity of $\{F \mid \mathbf{IL}X \vdash F\}$?
- Since **GL** $\subseteq$ **IL**, at least PSPACE for any natural choice of X.
- The only (?) known result: $\mathbf{IL}_0$ is PSPACE-hard.

- Our goal is to prove that **IL** is in PSPACE. The result might generalize to various **IL**X.

Complexity (2)

- Let F be any non-theorem of **IL**X. By completeness, there is $\mathcal{M}$, $w \in \mathcal{M}$ s.t. $w \nVdash F$.
  1. Show that $\mathcal{M}$ can be transformed to a certain model $\mathcal{M}^f$ with some desirable properties:
     - accessibility relation (R) is a tree;
     - polynomial height;
     - polynomial branching factor;
     - S_w-transitions should be "separated" or "factorized".
  2. Show that there is an algorithm that verifies the existence of all models with such properties. For **IL**X, ensure the resulting model has **IL**X.

Complexity (3)

- Transforming the model (step 1):
  1. Unravel the accessibility relation (R).
  2. Recursively apply the following operation to all nodes $x \in W$, starting from leaves:
     - 2.1 Denote $N = \{A \rhd B : w \nVdash A \rhd B\}$.
     - 2.2 Make $|N|$ copies of $R[x]$, label them with formulas of N.
     - 2.3 For all $A \rhd B \in N$, select a witness of falseness of $A \rhd B$ in the corresponding copy.
     - 2.4 In each copy, also select witnesses of true formulas.
  3. Remove worlds that do not witness anything.
- This leaves us with a model of polynomial height and branching factor; with easy to manage S_w relations.
- Step 2 is proving correctness and completeness of the algorithm that creates models resembling the ones from step 1.

Papers

- T. Perkov, M. Vuković. Filtrations of generalized Veltman models. *Mathematical Logic Quarterly*, **62**, 412–419, 2016.
- L. Mikec, T. Perkov, M. Vuković. Decidability of interpretability logics $\mathbf{IL}M_0$ and $\mathbf{IL}W^*$. Logic Journal of the IGPL, Volume 25, Issue 5, 1 October 2017, Pages 758–772,
- (complexity paper - work in progress)