Sound and complete subtyping on intersection and union types

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joint work with Mariangiola Dezani-Ciancaglini

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Subtyping

M. Dezani-Ciancaglini and SG.
Preciseness of subtyping on intersection and union types.
In *RTA-TLCA 2014*, volume 8560 of *LNCS*, pages 194–207 (2014).

M. Dezani-Ciancaglini, SG, S. Jakšić, J. Pantović and N. Yoshida.
Denotational and Operational Preciseness of Subtyping: A Roadmap.
In *Theory and Practice of Formal Methods 2016*, LNCS 9660: 155–172, 2016.

Subtyping

Subtyping is a binary relation $\leq$ (preorder) on the set of `Types`

$$\sigma \leq \tau$$

Subsumption rule in the type inference system

$$\frac{M : \sigma \quad \sigma \leq \tau}{M : \tau}$$

- λ-calculi, concurrent calculi
- programming languages

1. Intersection types and subtyping in λ-calculus
2. Soundness and completeness of subtyping
3. Concurrent λ-calculus
4. Preciseness Results
5. Related and further work

1. Intersection types and subtyping in λ-calculus

2. Soundness and completeness of subtyping

3. Concurrent λ-calculus

4. Preciseness Results

5. Related and further work

Intersection types

- The abstract grammar that generates the language

$$\sigma \;::=\; \alpha \mid \sigma \to \sigma \mid \sigma \cap \sigma$$

- Axiom

$$\frac{}{\Gamma, x : \sigma \vdash x : \sigma}\,(Ax)$$

- Rules

$$\frac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}\,(elim \to) \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau}\,(intr \to)$$

$$\frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \sigma}\,(elim\cap) \qquad \frac{\Gamma \vdash M : \sigma \cap \tau}{\Gamma \vdash M : \tau}\,(elim\cap) \qquad \frac{\Gamma, \vdash M : \sigma \quad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \cap \tau}\,(intr\cap)$$

$$\frac{\Gamma, \vdash M : \sigma \quad \sigma \leq \tau}{\Gamma \vdash M : \tau}\,(\leq)$$

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$$\frac{\Gamma, \vdash M : \sigma \quad \sigma \leq \tau}{\Gamma \vdash M : \tau} \, (\leq)$$

$\lambda \rightarrow$ vs $\lambda\cap$

$\lambda \rightarrow$

- M is typable $\Longrightarrow$ M is SN
- Curry-Howard correspondence formulae-as-types, proofs-as-terms, proofs-as programs
- M :?, typability is decidable
- ? : σ, inhabitation is decidable
- $(M : \sigma)$?, type checking is decidable
- $\lambda x.xx$: *NO*

$\lambda\cap$

- M is typable $\Longleftrightarrow$ M is SN
- Filter models based on subtyping
- $\lambda x.xx : ((\sigma \rightarrow \tau) \cap \sigma) \rightarrow \tau$
- NO Curry-Howard
- Typability, inhabiatation, type checking - undecidable

$\lambda \to$ vs $\lambda\cap$

$\lambda \to$

- M is typable $\Longrightarrow$ M is SN
- Curry-Howard correspondence formulae-as-types, proofs-as-terms, proofs-as programs
- M :?, typability is decidable
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Subtyping - preodrer

1. $\sigma \leq \sigma$ (reflexive)
2. $\sigma \leq \tau,\ \tau \leq \gamma\ \Rightarrow \sigma \leq \gamma$ (transitive)

3. $\sigma \cap \tau \leq \sigma,\ \sigma \cap \tau \leq \tau$
4. $\sigma \leq \tau,\ \sigma \leq \gamma \Rightarrow \sigma \leq \tau \cap \gamma$

5. $\sigma \leq \sigma',\ \tau \leq \tau' \Rightarrow \sigma \cap \tau \leq \sigma' \cap \tau'$
6. $\sigma \leq \sigma',\ \tau \leq \tau' \Rightarrow \sigma' \rightarrow \tau \leq \sigma \rightarrow \tau'$

7. $(\sigma \rightarrow \tau) \cap (\sigma \rightarrow \gamma) \leq \sigma \rightarrow \tau \cap \gamma$

8. $\sigma \leq \Omega$
9. $\sigma \rightarrow \Omega \leq \Omega \rightarrow \Omega$.

The induced equivalence relation:

$\sigma \sim \tau \Leftrightarrow \sigma \leq \tau\ \&\ \tau \leq \sigma$. (symmetric)

Preciseness of subtyping

Preciseness

- Soundness
- Completeness

Two aspects:

- Denotational preciseness
- Operational preciseness

Denotational Preciseness of Subtyping

$[\![\sigma]\!]$ is a set interpreting type σ

denotational soundness: $\sigma \leq \tau$ implies $[\![\sigma]\!] \subseteq [\![\tau]\!]$

denotational completeness: $[\![\sigma]\!] \subseteq [\![\tau]\!]$ implies $\sigma \leq \tau$

denotational preciseness: $\sigma \leq \tau$ iff $[\![\sigma]\!] \subseteq [\![\tau]\!]$

H. Barendregt, M. Coppo, and M. Dezani-Ciancaglini.
A Filter Lambda Model and the Completeness of Type Assignment.
Journal of Symbolic Logic, 48(4):931–940, 1983.

J. Vouillon.
Subtyping Union Types.
In *CSL*, volume 3210 of *LNCS*, pages 415–429, 2004.

Operational Soundness of Subtyping

If $\sigma \leq \tau$, **then** each context

- that is safe when filled with a term of type τ is also safe when filled with a term of type σ

$$\forall C[\,]\ (\forall M : \tau\ C[M] \not\to^* \texttt{error} \implies \forall N : \sigma\ C[N] \not\to^* \texttt{error})$$

Example. $\texttt{nat} \leq \texttt{int}$ $C[-5]$ converges, then $C[2]$ converges

Safe replacement

Operational soundness of subtyping follows from subject reduction of the type system with the subsumption rule

Operational Completeness of Subtyping

Converse:
If each context that is safe when filled with a term of type τ is also safe when filled with a term of type σ, **then** $\sigma \leq \tau$

Instead:
If $\sigma \not\leq \tau$, **then** there is a context

- that is safe when filled with an arbitrary term of type τ, and
- gives an error when filled with a suitable term of type σ

$$\exists C_0[\,](\forall M : \tau.\ C_0[M] \not\to^* \mathtt{error} \bigwedge \exists N_0 : \sigma.C_0[N_0] \to^* \mathtt{error})$$

J. Blackburn, I. Hernandez, J. Ligatti, and M. Nachtigal.
Completely subtyping iso-recursive types.
Technical Report, University of South Florida, 2014.

Concurrent λ-calculus - Syntax

M. Dezani-Ciancaglini, U. de'Liguoro, and A. Piperno.
A Filter Model for Concurrent Lambda-Calculus.
SIAM Journal on Computing 27(5):1376–1419, 1998.

$$M ::= x \mid v \mid (\lambda x.M) \mid (\lambda v.M) \mid (MM) \mid (M + M) \mid (M \| M)$$

1. call-by-name and call-by-value variables
2. internal choice
3. parallel operator

$$W ::= v \mid \lambda x.M \mid \lambda v.M \mid W \| W$$
$$V ::= W \mid V \| M \mid M \| V$$

TVal total values:
Val values

Reduction rules

$$(+_L)\ M + N \longrightarrow M \qquad (+_R)\ M + N \longrightarrow N$$

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$$(+_L)\ M + N \longrightarrow M \qquad (+_R)\ M + N \longrightarrow N$$

$$(\|_{app})\ (M\|N)L \longrightarrow ML\|NL \qquad (\|_s)\ \frac{M \longrightarrow M' \quad N \longrightarrow N'}{M\|N \longrightarrow M'\|N'}$$

$$(\|_a)\ \frac{M \longrightarrow M' \quad W \in \text{TVal}}{M\|W \longrightarrow M'\|W,\ W\|M \longrightarrow W\|M'}$$

TVal *total values*: $W ::= v \mid \lambda x.M \mid \lambda v.M \mid W\|W$

Reduction rules

$$(+_L)\ M + N \longrightarrow M \qquad (+_R)\ M + N \longrightarrow N$$

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$$(\|_a)\ \frac{M \longrightarrow M' \quad W \in \mathrm{TVal}}{M\|W \longrightarrow M'\|W,\ W\|M \longrightarrow W\|M'}$$

$$(\beta)\ (\lambda x.M)N \longrightarrow M[N/x] \qquad (\beta_v)\ \frac{W \in \mathrm{TVal}}{(\lambda v.M)W \longrightarrow M[W/v]}$$

$$(\beta_v\|)\ \frac{V \longrightarrow V' \quad V \in \mathrm{Val}}{(\lambda v.M)V \longrightarrow M[V/v]\|(\lambda v.M)V'}$$

TVal *total values*: $W ::= v \mid \lambda x.M \mid \lambda v.M \mid W\|W$

Val *values* $V ::= W \mid V\|M \mid M\|V$

Reduction rules

$$(+_L)\ M+N \longrightarrow M \qquad (+_R)\ M+N \longrightarrow N$$

$$(\|_{app})\ (M\|N)L \longrightarrow ML\|NL \qquad (\|_s)\ \frac{M \longrightarrow M' \quad N \longrightarrow N'}{M\|N \longrightarrow M'\|N'}$$

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$$(\beta_v\|)\ \frac{V \longrightarrow V' \quad V \in \mathrm{Val}}{(\lambda v.M)V \longrightarrow M[V/v]\|(\lambda v.M)V'}$$

$$(\mu_v)\ \frac{N \longrightarrow N' \quad N \notin \mathrm{Val}}{(\lambda v.M)N \longrightarrow (\lambda v.M)N'} \qquad (\nu)\ \frac{M \longrightarrow M' \quad M \notin \mathrm{Val} \bigcup \mathrm{Par}}{MN \longrightarrow M'N}$$

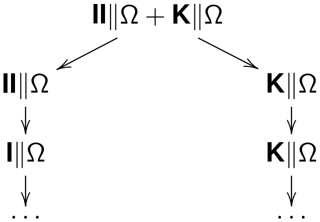
TVal *total values*: $W ::= v \mid \lambda x.M \mid \lambda v.M \mid W\|W$

Val *values* $V ::= W \mid V\|M \mid M\|V$

$\mathrm{Par} = \{M\|N\}$

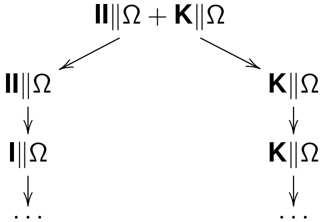
Convergence

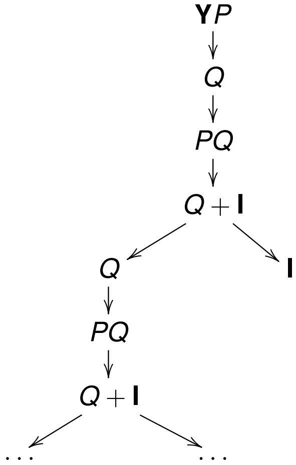
reduction tree



Convergence

reduction tree $\quad P = \lambda x.(x + \mathbf{I}) \quad Q = (\lambda x.P(xx))(\lambda x.P(xx))$

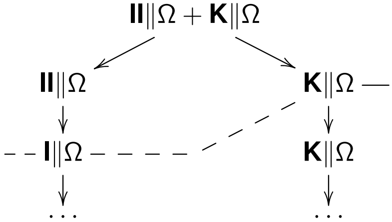


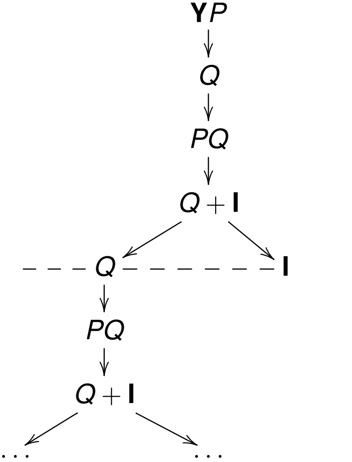


Convergence

reduction tree $\qquad P = \lambda x.(x + \mathbf{I}) \quad Q = (\lambda x.P(xx))(\lambda x.P(xx))$

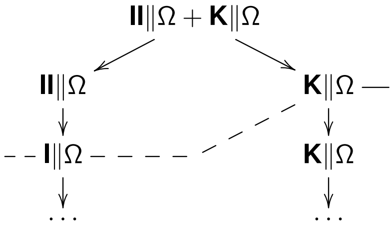
Bar is a subset of nodes of the reduction tree such that each maximal path intersects the bar at exactly one node

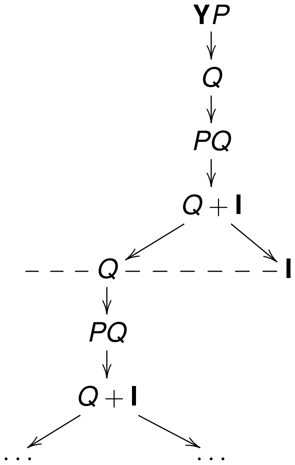




Convergence

a term converges if there is a bar of values in its reduction tree





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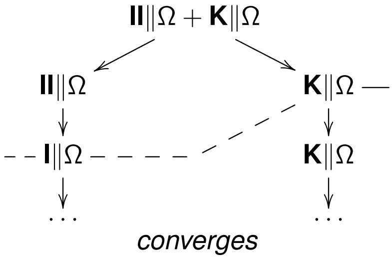

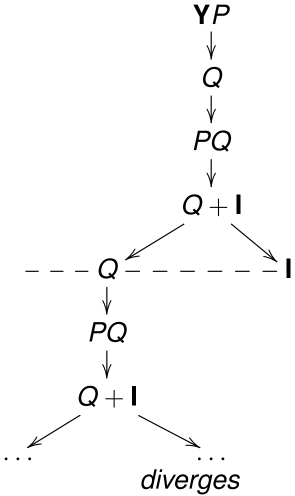

Types and Subtyping

$\texttt{Type}$: $\sigma ::= \omega \mid \sigma \to \sigma \mid \sigma \wedge \sigma \mid \sigma \vee \sigma$

$\sigma \leq \tau$ is the smallest pre-order on types such that

1. $\langle \texttt{Type}, \leq \rangle$ is a distributive lattice, in which $\wedge$ is the meet, $\vee$ is the join and ω is the top;
2. the arrow satisfies
   1. $\sigma \to \omega \leq \omega \to \omega$;
   2. $(\sigma \to \rho) \wedge (\sigma \to \tau) \leq \sigma \to \rho \wedge \tau$;
   3. $\sigma \geq \sigma', \tau \leq \tau' \Rightarrow \sigma \to \tau \leq \sigma' \to \tau'$.

$\texttt{CType}$: a type σ is coprime if $\sigma \leq \tau \vee \rho$ implies $\sigma \leq \tau$ or $\sigma \leq \rho$

Each type is equal to a union of coprime types.

Typing Rules

A basis Γ maps

1. call-by-name variables to types (ω by default) and
2. call-by-value variables to coprime types ($\omega \to \omega$ by default)

Typing Rules

$$(\mathrm{Ax})\ \Gamma \vdash \alpha : \Gamma(\alpha)$$

Typing Rules

$$(\mathrm{Ax})\ \Gamma \vdash \alpha : \Gamma(\alpha) \qquad (\omega)\ \Gamma \vdash M : \omega$$

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$$(\to \mathrm{I}_n)\ \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau}$$

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$$(\to \mathrm{I}_v)\ \frac{\Gamma, v : \sigma_i \vdash M : \tau \quad \sigma = \bigvee_{i \in I} \sigma_i \quad \sigma_i \in \mathtt{CType}\ i \in I}{\Gamma \vdash \lambda v.M : \sigma \to \tau}$$

Typing Rules

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$$(\to \mathrm{E})\ \frac{\Gamma \vdash M : \sigma \to \tau\ \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$$

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$$(\wedge \mathrm{I})\ \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash M : \tau}{\Gamma \vdash M : \sigma \wedge \tau}$$

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$$(+\mathrm{I})\ \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \tau}{\Gamma \vdash M + N : \sigma \vee \tau}$$

Typing Rules

$$(\mathrm{Ax})\ \Gamma \vdash \alpha : \Gamma(\alpha) \qquad (\omega)\ \Gamma \vdash M : \omega$$

$$(\to \mathrm{I}_n)\ \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau}$$

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$$(+\mathrm{I})\ \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \tau}{\Gamma \vdash M + N : \sigma \vee \tau} \qquad (\| \mathrm{I})\ \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \tau}{\Gamma \vdash M \| N : \sigma \wedge \tau}$$

Characterisation of Convergence

Each type is either a subtype of $\omega \to \omega$ or it is equivalent to ω.

Theorem (Type preservation)

The type system enjoys subject reduction.

Theorem

A closed term is convergent iff it has type $\omega \to \omega$.

Corollary

A closed term is divergent iff it has only types equivalent to ω.

Unsoundness of $(\sigma \to \rho) \wedge (\tau \to \rho) \leq \sigma \vee \tau \to \rho$

$\sigma = \rho \to \omega \to \rho \qquad \tau = \omega \to \rho \to \rho \qquad \rho = \omega \to \omega$

$\vdash \lambda x.x\mathbf{I}\Omega \| \lambda x.x\Omega\mathbf{I} : (\sigma \to \rho) \wedge (\tau \to \rho)$ and $\vdash \mathbf{K} + \mathbf{O} : \sigma \vee \tau$

If $(\sigma \to \rho) \wedge (\tau \to \rho) \leq \sigma \vee \tau \to \rho$ holds

$\vdash (\lambda x.x\mathbf{I}\Omega \| \lambda x.x\Omega\mathbf{I})(\mathbf{K} + \mathbf{O}) : \rho \;\; (= \omega \to \omega)$

$$\begin{aligned}(\lambda x.x\mathbf{I}\Omega \| \lambda x.\Omega\mathbf{I})(\mathbf{K} + \mathbf{O}) &\longrightarrow (\mathbf{K} + \mathbf{O})\mathbf{I}\Omega \| (\mathbf{K} + \mathbf{O})\Omega\mathbf{I}) \\ &\longrightarrow \mathbf{OI}\Omega \| \mathbf{K}\Omega\mathbf{I} \longrightarrow \Omega \| \Omega\end{aligned}$$

$\Omega \| \Omega$ diverges $\qquad \not\vdash \Omega \| \Omega : \omega \to \omega \qquad$ subject reduction fails!

Denotational preciseness for the Concurrent λ-calculus

The subtyping $\leq$ is denotationally precise when

$$\sigma \leq \tau \ \ \textbf{iff} \ \ [\![\sigma]\!] \subseteq [\![\tau]\!]$$

Theorem

The subtyping $\leq$ is denotationally precise for the concurrent λ-calculus.

$$[\![\sigma]\!] = \{M \mid \vdash M : \sigma\}$$

Operational preciseness for the Concurrent λ-calculus

The subtyping $\leq$ is operationally precise when

$\sigma \leq \tau$ **iff** for every closed term *M*
that *converges* when applied to a term of type τ also *converges* when applied to a term of type σ

$$\forall M (\forall P : \tau.\ MP \text{ conveges } \bigwedge \forall N : \sigma.\ MN \text{ converges })$$

Theorem

The subtyping $\leq$ is operationally precise for the concurrent λ-calculus.

Operational preciseness - general methodology

Operational soundness follows immediately from

- the subject reduction theorem,
- the subsumption rule, where the subtyping is used

Operational preciseness - general methodology

A general methodology to prove operational completeness is the following one:

- **[Step 1]** Characterise the negation of the subtyping relation by inductive rules
- **[Step 2]** For each type σ define a characteristic context C_σ, which behaves well when filled with terms of type σ
- **[Step 3]** For each type σ define a characteristic term M_σ, which has only the types greater than or equal to σ
- **[Step 4]** Show that if $\sigma \not\leq \tau$, then bad$(C_\tau[M_\sigma])$

Related work

J. Blackburn, I. Hernandez, J. Ligatti, and M. Nachtigal.
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Preciseness for Pure λ-Calculus

Operational completeness requires that all empty (i.e. not inhabited) types are less than all inhabited types

Inhabitation is undecidable for intersection types and for polymorphic types

A complete subtyping on intersection types or on polymorphic types for the pure λ-calculus must be undecidable

This makes unfeasible an operationally complete subtyping for the pure λ-calculus, both in case of intersection and union types and polymorphic types

The terms of the concurrent λ-calculus inhabit all types

Open problem: to study the extensions of λ-calculus enjoying operational preciseness for the decidable subtyping between polymorphic types