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CONSTRUCTION OF MEASURE-TYPE FUNCTIONS USING AGGREGATION FUNCTIONS[1]

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Abstract. This paper presents the construction of a new measure-type functions of a certain type, by applying aggregation functions on the initial sequence of measure-type functions of the same type. The measure-type functions considered in this paper are distance functions, metrics, fuzzy metrics, and fuzzy measures. The properties of the constructed new measure-type functions depend on the properties of applied aggregation functions, as well as the properties of the initial functions on which the aggregation function is applied.

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1. Introduction

In this review paper, some results of research conducted in the past ten years are presented, on the subject of the construction of new measure-type functions in the most wide sense, see [3, 4, 5]. The topic of the mentioned research is the methodology of construction of new measure-type functions by applying aggregation function on a sequence of initial functions of the same type. In this paper, considered measure-type functions are

(a) distance functions and metrics in the Section 3, see [4],

(b) fuzzy metrics in the Section 4, see [5],

(c) fuzzy measures in the Section 5, see [3].

The definition and some relevant properties of aggregation functions are presented in Section 2.

The idea of the mentioned methodology consists in the following. Let I be one of the intervals $I_1 = [0, 1]$, $I_2 = [0, \infty)$ or $I_3 = [0, \infty]$, and let $n \in \mathbb{N}$. Let X

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be an arbitrary nonempty set measurable in a certain sense, and let $m_i : X \to I$, $i \in \{1, \ldots, n\}$ be a sequence of some measure-type functions of the same type. For n-ary aggregation function $A_{[n]} : I^n \to I$, function $m_{[n]} : X \to I$ defined with

$$m_{[n]}(x) = A_{[n]}\left(m_1(x), \ldots, m_n(x)\right), \quad x \in X \tag{1.1}$$

is also a measure-type function of the same type as the functions $m_i : X \to I$, $i \in \{1, \ldots, n\}$. Properties of the function m depend on the properties of the initial functions $m_i : X \to I$, $i \in \{1, \ldots, n\}$, as well as on the properties of applied aggregation functions $A_{[n]}$. Let I be one of the intervals $I_1 = [0, 1]$, $I_2 = [0, \infty)$ or $I_3 = [0, \infty]$ throughout the whole paper.

2. Aggregation functions

Aggregation functions are one type of fuzzy-operations that have found significant use in IT and other engineering disciplines, but also in other sciences. There are numerous types, classes, additional possible properties and methods of construction of aggregation functions, see [2]. Various sets of argument values and aggregation function values are discussed in the literature. In this article, the intervals I specified in Section 1 are considered as the mentioned sets.

Definition 2.1 (Aggregation function)**.** For each $n \in \mathbb{N}$, an *n-ary aggregation function* is a function $A_{[n]} : I^n \to I$ with the following properties.

(a01) *Boundary conditions* hold, which particulary means that

$$A_{[n]}(0, \ldots, 0) = 0, \tag{2.1}$$

and, depending on the observed cases for interval I,

[I1] for $I = I_1 = [0, 1]$,

$$A_{[n]}(1, \ldots, 1) = 1, \tag{2.2}$$

[I2] for $I = I_2 = [0, \infty)$,

$$\lim_{\forall i \in \{1,\ldots,n\},\ a_i \to \infty} A_{[n]}(a_1, \ldots, a_n) = \infty, \tag{2.3}$$

[I3] for $I = I_3 = [0, \infty]$,

$$A_{[n]}(\infty, \ldots, \infty) = \infty. \tag{2.4}$$

(a02) A function A is monotonically non-decreasing in each component, i.e., implication

$$\forall i \in \{1, \ldots, n\},\ a_i \le b_i \quad \Rightarrow \quad A_{[n]}(a_1, \ldots, a_n) \le A_{[n]}(b_1, \ldots, b_n) \tag{2.5}$$

hold for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in I^n$.

For $n = 1$, by definition is $A_{[1]}(x) = x$, $x \in I$.

Some additional properties that aggregation functions can have, can be of interest in the consideration of the construction of mentioned measure-type functions.

Definition 2.2. An n-ary aggregation function $A_{[n]} : I^n \to I$ can have some of the following properties.

(a03) Function $A_{[n]}$ is *continuous*.

(a04) Function $A_{[n]}$ is *symmetric in each component*, i.e., for each n-tuple $(a_1, \ldots, a_n) \in I^n$ and each permutation p of the set $\{1, \ldots, n\}$ hold

$$A_{[n]}(a_1, \ldots, a_n) = A_{[n]}(a_{p(1)}, \ldots, a_{p(n)}).$$

(a05) Function $A_{[n]}$ is *idempotent*, i.e., for each $a \in I$ holds $A_{[n]}(a, \ldots, a) = a$.

(a06) Function $A_{[n]}$ is *subadditive*, i.e., for all n-tuples $(a_1, \ldots, a_n) \in I^n$ and $(b_1, \ldots, b_n) \in I^n$ that satisfy condition $(a_1 + b_1, \ldots, a_n + b_n) \in I^n$ holds

$$A_{[n]}(a_1 + b_1, \ldots, a_n + b_n) \le A_{[n]}(a_1, \ldots, a_n) + A_{[n]}(b_1, \ldots, b_n).$$

(a07) Function $A_{[n]}$ is *superadditive*, i.e., for all n-tuples $(a_1, \ldots, a_n) \in I^n$ and $(b_1, \ldots, b_n) \in I^n$ that satisfy condition $(a_1 + b_1, \ldots, a_n + b_n) \in I^n$ holds

$$A_{[n]}(a_1 + b_1, \ldots, a_n + b_n) \ge A_{[n]}(a_1, \ldots, a_n) + A_{[n]}(b_1, \ldots, b_n).$$

(a08) Function $A_{[n]}$ is *positively homogeneous*, i.e., for each $t \ge 0$ and all n-tuples $(a_1, \ldots, a_n) \in I^n$ that satisfy condition $(ta_1, \ldots, ta_n) \in I^n$ holds

$$A_{[n]}(ta_1, \ldots, ta_n) \le tA_{[n]}(a_1, \ldots, a_n).$$

(a09) Depending on interval I, for $A_{[n]}$ holds one of following implications.

[I1] For $I = I_1 = [0, 1]$,

$$A_{[n]}(a_1, \ldots, a_n) < 1 \;\Rightarrow\; \forall i \in \{1, \ldots, n\},\ a_i < 1.$$

[I2] For $I = I_2 = [0, \infty)$ and $(b_1, \ldots, b_n) \in [0, \infty]^n$,

$$\lim_{\forall i \in \{1,\ldots,n\},\ a_i \to b_i} A_{[n]}(a_1, \ldots, a_n) < \infty \;\Rightarrow\; \forall i \in \{1, \ldots, n\},\ b_i < \infty.$$

[I3] For $I = I_3 = [0, \infty]$,

$$A_{[n]}(a_1, \ldots, a_n) < \infty \;\Rightarrow\; \forall i \in \{1, \ldots, n\},\ a_i < \infty.$$

(a10) Function $A_{[n]}$ satisfies

$$A_{[n]}(a_1, \ldots, a_n) = 0 \;\Rightarrow\; \exists i \in \{1, \ldots, n\},\ a_i = 0.$$

(a11) Function $A_{[n]}$ satisfies

$$A_{[n]}(a_1, \ldots, a_n) = 0 \;\Rightarrow\; \forall i \in \{1, \ldots, n\},\ a_i = 0.$$

3. Construction of distance functions

Distance functions and metrics have significant role and application in various scientific disciplines. Distance functions are functions that satisfy only two simple axioms, see [1].

Definition 3.1. Let X be an arbitrary nonempty set. *Distance function* on the set X is a function $d : X^2 \to [0, \infty)$ which has the following properties:

(d01) *reflexivity*, i.e.,

$$\forall x \in X, \;\; d(x, x) = 0,$$

(d02) *symmetry*, i.e.,

$$\forall x, y \in X, \;\; d(x, y) = d(y, x).$$

A *distance space* (X, d) is a set X equipped with a distance function d.

Some additional properties that distance functions can have can be of interest in various particular applications.

Definition 3.2. Let $X \neq \emptyset$, and let $d : X^2 \to [0, \infty)$ be a distance function on X. The function d may have some of the following additional properties.

(d03) *Identity of indiscernibles*, i.e.,

$$\forall x, y \in X, \;\; d(x, y) = 0 \;\Rightarrow\; x = y.$$

(d04) *Triangle inequality*, i.e.,

$$\forall x, y, z \in X, \;\; d(x, z) \leq d(x, y) + d(y, z).$$

(d05) *Ultrametric inequality*, i.e.,

$$\forall x, y, z \in X, \;\; d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

(d06) *C-triangle inequality*, i.e.,

$$\exists C \in [1, \infty), \; \forall x, y, z \in X, \;\; d(x, y) \leq C\,(d(x, z) + d(z, y)).$$

(d07) *Boundedness*, i.e.,

$$d : X^2 \to [0, 1] \;\;\vee\;\; \exists a > 0, \; d : X^2 \to [0, a].$$

Distance functions are one of measure-type function. In accordance with formula (1.1), starting from the initial distance functions, we can construct a new distance function as follows.

Let $n \geq 2$. Let $A_{[n]} : [0, 1]^n \to [0, 1]$ be an arbitrary n-ary aggregation function, and let $d_i : X^2 \to [0, 1]$, $i \in \{1, \ldots, n\}$ be a sequence of bounded distance functions on the nonempty set X. Let we observe the function $d_{[n]} : X^2 \to [0, 1]$ defined by

$$d_{[n]}(x, y) = A_{[n]}\left(d_1(x, y), \ldots, d_n(x, y)\right), \quad x, y \in X. \tag{3.1}$$

Theorem 3.3. *For functions $d_{[n]}$ defined with* (3.1)*, the following is valid.*

(a) *Function $d_{[n]}$ is a distance function.*

(b) *If for each of the distance functions* d_i*,* $i \in \{1, \ldots, n\}$ *the "identity of indiscernibles"* (d03) *holds and aggregation function* $A_{[n]}$ *has the property* (a10)*, then for function* $d_{[n]}$*, the "identity of indiscernibles"* (d03) *is also valid.*

(c) *If for at least one of the considered distance functions* d_i*,* $i \in \{1, \ldots, n\}$ *"identity of indiscernibles"* (d03) *holds and if the aggregation function* $A_{[n]}$ *has the property* (a11)*, then for the distance function* $d_{[n]}$ *"identity of indiscernibles"* (d03) *also holds.*

(d) *Let each of the distance functions* d_i*,* $i \in \{1, \ldots, n\}$ *be metric (satisfies* (d03) *and* (d04)*). Let* $A_{[n]} : [0, \infty)^n \to [0, \infty)$ *be subadditive function (satisfies* (a06)*) whose restriction on the set* $[0, 1]^n$ *is an aggregation function which also satisfies the property* (a10)*), then the distance function* $d_{[n]}$ *is a metric also.*

(e) *Assume that for each of the distance functions* d_i*,* $i \in \{1, \ldots, n\}$ *the C-triangle inequality* (d06) *holds. Let* $A_{[n]} : [0, \infty)^n \to [0, \infty)$ *be subadditive and positively homogeneous function (satisfies* (a06) *and* (a08) *properties) whose restriction on the set* $[0, 1]^n$ *is an aggregation function. Then for the distance function* $d_{[n]}$ *the C-triangle inequality* (d06) *also holds.*

4. Construction of fuzzy metrics

The definitions of fuzzy metrics are not unanimously accepted in the literature. For the purpose of presented results and terms in this paper see [5].

Definition 4.1. Let $X \neq \emptyset$.

(fS) For the continuous t-conorm $S : [0, 1]^2 \to [0, 1]$ and for the fuzzy set $s : X^2 \times (0, \infty) \to [0, 1]$, the ordered triple (X, s, S) is a *fuzzy S-metric space* and fuzzy set s *is a fuzzy S-metric* if for all $x, y, z \in X$ and all $\alpha, \beta > 0$ satisfies the following conditions.

(1) $s(x, y, \alpha) \in [0, 1)$,

(2) $s(x, y, \alpha) = 0 \Leftrightarrow x = y$,

(3) $s(x, y, \alpha) = s(y, x, \alpha)$,

(4) $S\left(s(x, y, \alpha), s(y, z, \beta)\right) \geq s(x, z, \alpha + \beta)$,

(5) Function $\overline{s} : (0, \infty) \to [0, 1]$, $\overline{s}(t) = s(x, y, t)$ is a continuous function.

(fT) For the continuous t-norm $T : [0, 1]^2 \to [0, 1]$ and for the fuzzy set $t : X^2 \times (0, \infty) \to [0, 1]$, the ordered triple (X, t, T) is a *fuzzy T-metric space* and fuzzy set t *is a fuzzy T-metric* if for all $x, y, z \in X$ and all $\alpha, \beta > 0$ satisfies the following conditions.

(1) $t(x, y, \alpha) \in (0, 1]$,

(2) $t(x, y, \alpha) = 1 \Leftrightarrow x = y$,

(3) $t(x, y, \alpha) = t(y, x, \alpha)$,

(4) $T(t(x, y, \alpha), t(y, z, \beta)) \le t(x, z, \alpha + \beta)$,

(5) Function $\bar{t} : (0, \infty) \to [0, 1]$, $\bar{t}(t) = t(x, y, t)$ is a continuous function.

In the following theorem, according the formula (1.1) is presented a method for a fuzzy S-pseudo metric and fuzzy T-pseudo metric construction by application of aggregation function on the sequence of initial fuzzy S-pseudo metrics, i.e., initial fuzzy T-pseudo metrics, see [5]. For a definitions of aggregation function which is continuously compatible with continuous t-conorms $S_1, \ldots, S_n$ with respect to a continuous t-conorm S, and aggregation function which is continuously compatible with continuous t-norms $T_1, \ldots, T_n$ with respect to a continuous t-norm T see [5].

Theorem 4.2. *Let $A_{[n]}$ be a continuous n-ary aggregation function.*

(afS) *Let aggregation function $A_{[n]}$ be continuously compatible with continuous t-conorms $S_1, \ldots, S_n$ with respect to a continuous t-conorm S. If the functions $s_i : X_i^2 \times (0, \infty) \to [0, 1]$, $i \in \{1, \ldots, n\}$ are fuzzy S-pseudo metrics on $X_i \neq \emptyset$, $i \in \{1, \ldots, n\}$ with respect to the t-conorms S_i, $i \in \{1, \ldots, n\}$ respectively, then for $X = X_1 \times \cdots \times X_n$, function $s : X^2 \times (0, \infty) \to [0, 1]$ defined with*

$$s(x, y, \alpha) = A_{[n]}\left(s_1(x_1, y_1, \alpha), \ldots, s_n(x_n, y_n, \alpha)\right), \tag{4.1}$$

$x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X$, $\alpha > 0$ is the fuzzy S-pseudo metric in a broader sense with respect to t-conorm S.

(afT) *Let aggregation function $A_{[n]}$ be continuously compatible with continuous t-norms $T_1, \ldots, T_n$ with respect to a continuous t-norm T. If the functions $t_i : X_i^2 \times (0, \infty) \to [0, 1]$, $i \in \{1, \ldots, n\}$ are fuzzy T-pseudo metrics on $X_i \neq \emptyset$, $i \in \{1, \ldots, n\}$ with respect to the t-norms T_i, $i \in \{1, \ldots, n\}$ respectively, then for $X = X_1 \times \cdots \times X_n$, function $t : X^2 \times (0, \infty) \to [0, 1]$ defined with*

$$t(x, y, \alpha) = A_{[n]}\left(t_1(x_1, y_1, \alpha), \ldots, t_n(x_n, y_n, \alpha)\right), \tag{4.2}$$

$x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X$, $\alpha > 0$ is the fuzzy T-pseudo metric in a broader sense with respect to t-norm T.

5. Construction of fuzzy measures

There are many definitions and types of fuzzy measures and non-additive measures, see [6]. In this paper, some types of fuzzy measures and their construction will be considered, see [3].

Definition 5.1. Let $\mathcal{A}$ be a σ-algebra on the set $X \neq \emptyset$. Function $m : \mathcal{A} \to I$ is a *fuzzy measure* on $\mathcal{A}$ if

(fm1) $m(\emptyset) = 0$,

(fm2) $\forall A, B \in \mathcal{A},\ \ A \subseteq B \ \Rightarrow\ m(A) \le m(B)$.

Fuzzy measures can have many other important properties.

Definition 5.2. Let $m : \mathcal{A} \to I$ be a fuzzy measure defined on σ-algebra $\mathcal{A}$ on the set X.

(fm3) m is *continuous from below* if for every family of sets $A_i \in \mathcal{A}$, $i \in \mathbb{N}$ nested as $A_1 \subseteq A_2 \subseteq \dots$ holds

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} m(A_i). \tag{5.1}$$

(fm4) m is *continuous from above* if for every family of sets $A_i \in \mathcal{A}$, $i \in \mathbb{N}$ nested as $A_1 \supseteq A_2 \supseteq \dots$ and satisfying that there exists $n_0 \in \mathbb{N}$ such that $m(A_{n_0}) < \infty$ holds

$$m\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} m(A_i). \tag{5.2}$$

(fm5) m is *subadditive* if for each disjoint pair of sets $A, B \in \mathcal{A}$ holds

$$m\left(A \cup B\right) \leq m(A) + m(B). \tag{5.3}$$

(fm6) m is *superadditive* if for each disjoint pair of sets $A, B \in \mathcal{A}$ holds

$$m\left(A \cup B\right) \geq m(A) + m(B). \tag{5.4}$$

For $n \geq 2$, let $A_{[n]} : I^n \to I$ be an n-ary aggregation function, and let $m_i : \mathcal{A} \to I$, $i \in \{1, \dots, n\}$ be a sequence of fuzzy measures on $\mathcal{A}$. In accordance with formula (1.1), let we observe the function $m_{[n]} : \mathcal{A} \to I$ defined by

$$m_{[n]}(A) = A_{[n]}\left(m_1(A), \dots, m_n(A)\right), \quad A \in \mathcal{A}. \tag{5.5}$$

Properties of function $m_{[n]}$ depends on properties of aggregation function $A_{[n]}$ and properties of initial fuzzy measures m_i.

Theorem 5.3. *Function $m_{[n]}$ defined by* (5.5) *is a fuzzy measure on $\mathcal{A}$. Additionally, the following statements hold.*

(a) *Let all fuzzy measures m_i, $i \in \{1, \dots, n\}$ be continuous from below. If $A_{[n]}$ is an continuous aggregation function, then fuzzy measure $m_{[n]}$ is also continuous from below.*

(b) *Let all fuzzy measures m_i, $i \in \{1, \dots, n\}$ be continuous from above. If $A_{[n]}$ be a continuous aggregation function and, in the case $I = [0, \infty]$, let $A_{[n]}$ additionally have property* (a09)*. Then fuzzy measure $m_{[n]}$ is also continuous from above.*

(c) *Let all fuzzy measures m_i, $i \in \{1, \dots, n\}$ be subadditive. If $A_{[n]}$ is a subadditive aggregation function, then fuzzy measure $m_{[n]}$ is also subadditive.*

(d) *Let all fuzzy measures m_i, $i \in \{1, \dots, n\}$ be superadditive. If $A_{[n]}$ is a superadditive aggregation function, then fuzzy measure $m_{[n]}$ is also superadditive.*

6. Conclusion

As for distance functions, fuzzy metrics, and fuzzy measures, construction proposed by (1.1) can be applied on other measure-type functions.

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