The 9th Conference on Mathematics in Engineering: Theory and Applications
Novi Sad, May 31st–June 2nd 2024

ON THE COMPUTATION OF THE ORTHOGONAL HULL OF SIMPLE RECTILINEAR POLYGONS

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https://doi.org/10.24867/META.2024.01 Review article

Abstract. We briefly review the algorithm for determining the orthogonal hull of a set of simple rectilinear polygons, proposed by Nicholl et al., based on determining their maximal vertices.

AMS Mathematics Subject Classification (2020): 68U05, 65D18

Key words and phrases: rectilinear polygons, orthogonal hull, orthogonal convexity

1. Introduction and preliminaries

A *simple polygon* is usually defined as a closed, connected plane figure determined by a cyclic sequence of points, called *(polygonal) vertices*, each one joined to the previous and the next point with straight line segments, and no other points of intersection exist between the line segments. The line segments themselves are then referred to as *(polygonal) edges*, and they form what is called the *(polygonal) boundary*, and outline a bounded region called the *(polygonal) area* or *(polygonal) interior*. See, for example, [9].

In [7], the authors consider only simple, rectilinear polygons (polygons whose interior angles are all $90°$ or $270°$) , which they call $X - Y$ polygons. They also refer to the two classes of edges such polygons possess as *vertical* or *horizontal*, and we adopt the same terminology. In what follows, we consider only simple rectilinear polygons, which we refer to simply as "polygons" for brevity.

Definition 1.1. A polygon is *h-convex* (*v-convex*) if each non-empty intersection of the polygonal interior with a horizontal (vertical) line is connected. If the polygon is both h-convex and v-convex, it is called *orthogonally convex* (or hv-convex).

Definition 1.2. [7, 8] The *orthogonal hull* (or $X - Y$ *convex hull*) of a set of polygons, if it exists, is an orthogonally convex polygon that contains the set of polygons and has minimal area.

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Note that the orthogonal hull of a set of polygons need not exist, as the polygon that contains the entire set and is of minimal area may not be simple at all, see Figure 1 for an example.

Figure 1: When considering a set of multiple polygons, the rectilinear polygon that contains all of them and is of minimal area: a) may be simple, and therefore the orthogonal hull, b) may not be simple, and is thus not the orthogonal hull.

In computer graphics, it is common to consider polygons as primitives and store the vertex coordinates and information regarding the connectivity of the edges [5]. Thus, for each polygon, we consider its vertices to be known and (their corresponding x- and y-coordinates) provided as the input for the algorithm.

Definition 1.3. [4, 10] Given a finite set of vertices, a point $p = (p_x, p_y)$ dominates (from the north-east direction) a distinct point $r = (r_x, r_y)$ if $r_x \leq p_x$ and $r_y \leq p_y$. A vertex p is maximal (from the north-east direction) if there is no other vertex from the set which dominates p.

The maximal vertices from the other three diagonal directions (north-west, south-east, south-west) are obtained in an analogous manner, see Figure 2.

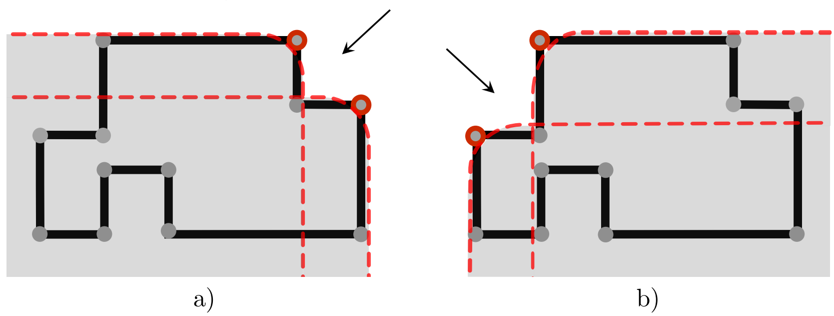

Figure 2: A polygon whose vertices are shown in grey, and its maximal vertices from a) north-east and b) north-west, shown outlined in red. Each maximal vertex dominates the vertices in its corresponding region (outlined with a red dashed line).

2. Algorithm by Nicholl et al.

In [7], the authors use as input a list of the polygonal vertices V, sorted in an order in which these vertices would have been visited during an oriented (clockwise) traversal of the polygonal boundary. We first examine the case when a single polygon's orthogonal hull is considered.

2.1. Determining the orthogonal hull for a single polygon

The goal of the algorithm is to find all maximal vertices among the polygonal vertices, as the orthogonal hull is uniquely determined by such vertices. Obviously, the four extremal edges (i.e., the topmost horizontal, leftmost horizontal, bottommost vertical and rightmost vertical edges) necessarily all belong to the orthogonal hull. They determine at most eight distinct maximal vertices. The rest of the maximal vertices can be found between:

1. the right topmost and top rightmost vertices (from north-east),
2. the bottom rightmost and right bottommost vertices (from south-east),
3. the left bottommost and bottom leftmost vertices (from south-west),
4. the top leftmost and left topmost vertices (from north-west).

Note that the coordinates of these eight maximal vertices can easily be obtained by simply going through the input list of vertices and checking their coordinates.

For simplicity, we describe the algorithm as if we were determining non-trivial maximal vertices between the right topmost and top rightmost vertices (i.e. from the north-east direction). The other three cases are analogous.

The algorithm uses the fact that, for the orthogonal hull to be traversed (clockwise) between the right topmost and top rightmost vertices, the only allowed directions of traversal are downwards and to the right, which determines a sequence of alternately vertical and horizontal edges, forming what the authors in [7] refer to as a *staircase*, see Figure 3.

Note, as well, that a clockwise traversal reaches these maximal vertices from the left, via horizontal edge, and continues traversal downwards from them via vertical edge. The authors in [7] refer to all such vertices (not only maximal ones) as *right-down convex*, see Figure 3.

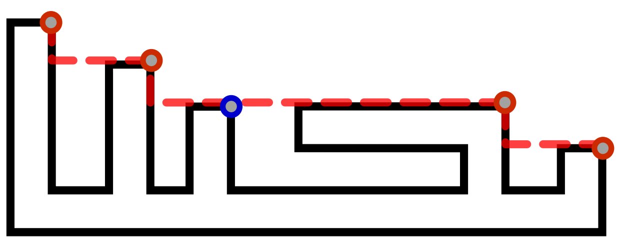

Figure 3: Maximal vertices (from the north-east direction), shown outlined in red, determine the part of the orthogonal hull from the right topmost to the top rightmost vertex of a given polygon (shown dashed in red). The alternately vertical and horizontal line segments connecting these vertices belong to the orthogonal hull. The only right-down convex vertex which is not maximal is shown outlined in blue.

The sorted list V_m of maximal vertices can be obtained as follows:

- At first, add the right topmost vertex to list V_m. It is necessarily a maximal (and right-down convex) vertex.
- After adding element v_i to V_m, find the first next right-down convex vertex v_{i+1} in V. Then:
  - if v_i dominates v_{i+1} (from north-east), examine the next right-down convex vertex in the same manner and do not add v_{i+1} to V_m;
  - if v_{i+1} does not dominate v_i, add v_{i+1} to V_m;
  - if v_{i+1} dominates v_i, remove all recent entries in V_m which are dominated by v_{i+1} (including v_i), and add v_{i+1} to V_m.
- If v_{i+1} is the top rightmost vertex, end the procedure. See Figure 4.

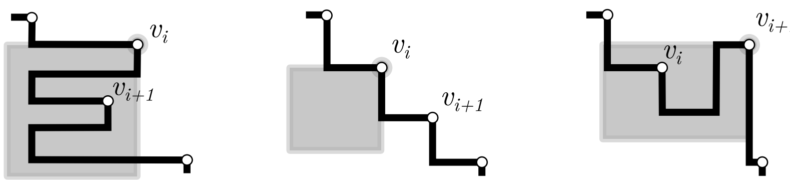

Figure 4: An example of traversal from the right topmost to the top rightmost vertex, with right-down convex vertices labeled v_i and v_{i+1}. Left: v_i dominates v_{i+1} (from north-east), center: v_{i+1} does not dominate v_i, right: v_{i+1} dominates v_i.

The output of the procedure described above returns a list of maximal vertices, which determine a part of the orthogonal hull. The maximal vertices in the remaining three directions may be found in an analogous manner. If a vertex is deleted at any point from the list of maximal vertices, it is not encountered again, nor added later. Thus, the total time complexity is $O(n)$, where n is the number of polygonal vertices, i.e. the size of the list V.

2.2. Determining the orthogonal hull for multiple polygons

The authors also describe how this algorithm could be adapted for the case of multiple polygons, provided they satisfy the condition of not having an $X-Y$ separation:

Definition 2.1. [7] The *x-extent* of a set of polygons is the open interval (x_{min}, x_{max}) where x_{min} and x_{max} are, respectively, the minimum and maximum x-coordinates among all vertices of the set. The *y-extent* is analogously defined.

Definition 2.2. [7] A set of polygons has an $X-Y$ separation if the set can be partitioned into disjoint nonempty subsets (of polygons) which have disjoint x-extents and y-extents.

Figure 1b) depicts a set of two polygons that has an $X-Y$ separation.

It is proven in [7] that the nonexistence of an $X - Y$ separation implies the existence of the orthogonal hull of a set of polygons. Thus, for a given set of k polygons which have no $X - Y$ separation, first the algorithm above is applied for each individual polygon, in order to find all maximal vertices. Then the maximal vertices forming, for example, the staircases connecting the right topmost to the top rightmost vertex of each individual polygon are also the candidates for forming the appropriate staircase of the entire orthogonal hull of the set.

These maximal vertices are then all lexicographically sorted, so that (x_1, y_1) comes before (x_2, y_2) if either: a) $x_1 > x_2$ or b) $x_1 = x_2$ and $y_1 > y_2$, and the sorted list is then traversed element by element. If an encountered element is dominated by the previous element in the sorted list, it is removed from the list. What remains is the list of maximal vertices that determine the required section of the orthogonal hull, see Figure 5. An analogous approach is used to find the remaining maximal vertices from the three other directions. The time complexity is $O(n \log k)$, where k is the number of polygons in the set, and n the total number of all polygonal vertices.

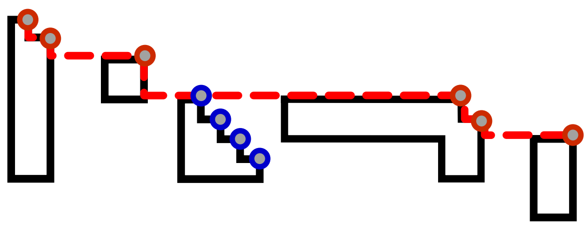

Figure 5: Maximal vertices (from the north-east direction), shown outlined in red and blue, for each individual polygon. Those outlined in blue will be removed, leaving only those which determine the required section of the orthogonal hull of the set (shown dashed in red).

3. Conclusion

The orthogonal hull of a digital object represents an important notion in computational geometry. It has found number of interesting applications tied to image processing and computer aided design, operations research, database concurrency control, fault-tolerant routing in mesh-connected multicomputers, as well as very large-scale integrated (VLSI) circuit layout design, see [1, 2, 3, 4, 6, 11], for instance. In this paper, we review the algorithm proposed by Nicholl et al. [7], which addresses the problem of determining the orthogonal hull for a single simple, rectilinear polygon, or a set of such polygons, and gives the condition for the existence of such a hull in the latter case.

Acknowledgement

This paper has been supported by the project “Improving the teaching process in the English language in fundamental disciplines”, Faculty of Technical Sciences, University of Novi Sad.

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