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A MODIFIED DURAN-SHISHKIN MESH FOR A SINGULARLY PERTURBED THIRD ORDER BOUNDARY VALUE PROBLEM[1]

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Abstract. In this article, a third order singularly perturbed problem with a weak layer is considered. To obtain approximation to the solution of this problem, a standard difference scheme on a new modification of the Duran-Shishkin mesh is used. Our modification has several advantages and provides numerical solution with better accuracy then the standard Shishkin mesh, which is confirmed in numerical experiments.

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1. Introduction

We consider the following third order singularly perturbed problem:

$$\begin{aligned} \varepsilon u'''(x) + a(x)u''(x) + b(x)u'(x) + c(x)u(x) &= f(x) \quad \text{in} \quad \Omega \\ u(0) = u(1) = u'(0) &= 0, \end{aligned} \tag{1.1}$$

where $\Omega = (0,1)$, ε is a small positive parameter, $a(x) > \alpha > 0$, and the coefficients are infinitely differentiable. This problem has a weak layer near $x = 0$ and from [6] we have

$$|u^{(k)}(x)| < C(1 + \varepsilon^{-k+1}e^{-\alpha x/\varepsilon}), \quad k = 0, 1, 2,$$

where C is some constant independent of ε.

There are only few papers dealing with third order singularly perturbed problems, see for example [3],[4],[6]-[8]. These problems are more difficult than second or fourth order singularly perturbed problem, which are studied widely.

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2. Discretization

For a given arbitrary mesh $0 = x_0 < x_1 < \ldots < x_n = 1$, the following notation is used

$$h_i = x_{i+1} - x_i, \quad i = 0, 1, \ldots, n-1, \qquad h = \max_{i=0,1,\ldots,n} h_i,$$

and the divided difference operator

$$D^k u_i^h := k! u^h[x_i, \ldots, x_{i+k}], \quad k = 1, 2, 3$$

for a grid function u^h. The following discretization

(2.1)
$$\begin{aligned} &\varepsilon D^3 u_i^h + a_i D^2 u_{i+1}^h + b_i D^1 u_{i+1}^h + c_i u_{i+1}^h = f_i, \quad \text{for} \quad i = 0, 1, \ldots, n-3, \\ &u_0^h = u_n^h = 0, \quad D^1 u_0^h = 0. \end{aligned}$$

is used to obtain approximate solution to problem (1.1), [6]. Also, to measure the error of the approximation, we introduce the following discrete norms

$$\|u^h\|_{h,\varepsilon} := \max\{\|u^h\|_{h,\infty}, \|D^1 u^h\|_{h,\infty}, \varepsilon\|D^2 u^h\|_{h,\infty}\},$$

$$\|D^k u^h\|_{h,\infty} := \max\{|D^k u_i^h| : i = 0, 1, \ldots, n-3\}, \quad k = 0, 1, 2.$$

Finally, $\|\cdot\|_{h,\infty}$ is the standard maximum discrete norm.

3. Duran-Shishkin mesh

To capture boundary layers for singularly perturbed problems, the layer-adapted meshes are most often used. One type of those meshes is the Duran mesh introduced in [1] and it is obtained as a simplified version of Gartland-type mesh from [2]. In [7] the authors presented a version of the Duran mesh, the so-called Duran-Shishkin (D-S) mesh which was used to capture the layers for a third order singularly perturbed problem.

Let $0 < \mu < 1$ be a given parameter and N some chosen even integer. Mesh from [7] tailored for problem (1.1) is defined as follows:

$$\begin{aligned} &x_0 = 0, \\ &x_1 = \mu\varepsilon, \\ &x_i = x_{i-1} + \mu x_{i-1} = \mu(1+\mu)^{i-1}\varepsilon, \quad && i = 2, \ldots, M, \\ &x_{M+i} = \bar{\tau} + 2iN^{-1}(1-\bar{\tau}), && i = 1, \ldots, N/2, \\ &x_{M+N/2} = 1, \end{aligned}$$

where $x_M = \bar{\tau}$, and M is the smallest integer such that

$$\bar{\tau} = \mu(1+\mu)^{M-1}\varepsilon \geq \tau := \min\left\{\frac{1}{2}, \frac{2}{\alpha}\varepsilon \ln N\right\}.$$

This mesh has two issues. The fist one is a transition point $\bar{\tau}$ which is not the standard transition point of the Shishkin mesh τ. The second issue is related to the number of mesh points, which is $M + N/2 + 1$. This number is not a priori known since it depends on the mesh parameter μ. Also, it is not clear what is the ratio between M - the number of mesh points in the layer, and $N/2$ - the number of mesh points outside of it. If μ is large, then this ratio could be too small (M is very small). If μ is small, than the ratio could be too large (M is very large number).

The main goal here is to modify this mesh, in order to solve both issues given the above. For a fixed N, our modified mesh is defined as follows

$$\begin{aligned} &x_0 = 0, \\ &x_1 = \mu\varepsilon, \\ &x_i = x_{i-1} + \mu x_{i-1} = \mu(1+\mu)^{i-1}\varepsilon, \quad i = 2, \dots, M, \\ &x_{N/2+j} = \tau + 2iN^{-1}(1-2\tau), \qquad i = 1, \dots, N/2, \end{aligned} \tag{3.1}$$

where $M = N/2$, and μ is calculated so that

$$x_M = \mu(1+\mu)^{M-1}\varepsilon = \tau. \tag{3.2}$$

Now, the transition point of such a mesh is exactly τ and it has the same number of mesh points inside and outside of boundary layer. Precisely, there are $N/2$ points in the interval $(0, \tau)$ and $N/2$ points in the interval $(\tau, 1)$.

On Figure 1 the position of mesh points (for different values of ε) is shown. With decreasing of parameter ε, the number of mesh points in the neighborhood of $x = 0$ increases.

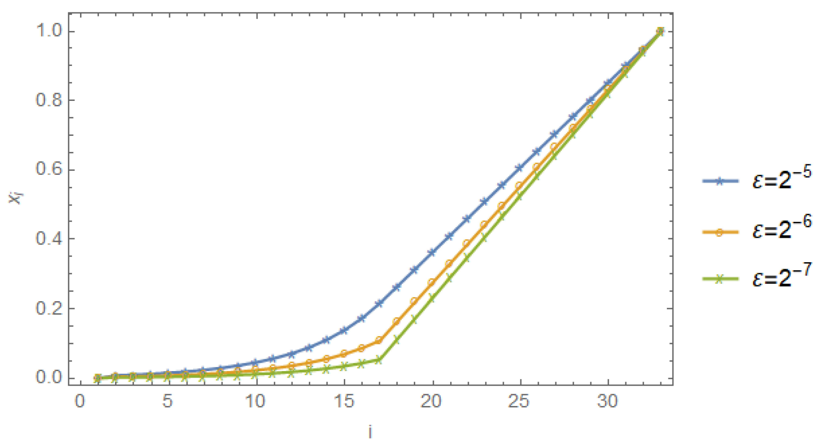

Figure 1: The modified D-S mesh (3.1) for $N = 32$.

The parameter μ in (3.2) can be calculated numerically. Table 1 shows the values of μ, for fixed ε and various N. Using these values, the obtained meshes are unique. In other words, for a chosen N, there exists only one modified D-S mesh, which is unlikely for the original D-S mesh [5], Duran mesh, as well for the one from [7].

For the method (2.1) applied on modified mesh (3.1), numerical solution converges to the exact solution of (1.1), when $N \to \infty$. Theoretical results

N	μ
2^5	0.2484575
2^6	0.1406596
2^7	0.0792910
2^8	0.0444235
2^9	0.0247098
2^{10}	0.0136418
2^{11}	0.0074769
2^{12}	0.0040706

Table 1: Values of parameter μ for $\varepsilon = 10^{-6}$

for different numerical methods on some kind of D-S mesh can be found in [6] and [7]. In this paper we want to compare numerical results for method (2.1) on mesh (3.1) with the results obtained by the same method on the standard piecewise uniform Shishkin mesh, which is defined by

$$x_i = \begin{cases} i\dfrac{2}{N}\tau, & i = 0, 1, \ldots, N/2 \\ \tau + \left(i - \dfrac{N}{2}\right)(1-\tau)\dfrac{2}{N}, & i = N/2+1, N/2+2, \ldots, N. \end{cases}$$

4. Numerical results

The following test problem is used to compare efficiency of method (1.1) on two mentioned

$$\begin{aligned} \varepsilon u'''(x) + u''(x) + \varepsilon u'(x) + u(x) &= x \quad \text{in} \quad \Omega = (0,1), \\ u(0) = u(1) = u'(0) = 0, \end{aligned}$$

The exact solution of this problem is given by

$$\begin{aligned} u(x) = \frac{\left(e^{\frac{1}{\varepsilon}}(\varepsilon - 1) - \varepsilon\right)\sin x}{\varepsilon + e^{\frac{1}{\varepsilon}}(\sin 1 - \varepsilon \cos 1)} &+ \frac{\varepsilon\left(e^{\frac{1}{\varepsilon}}(1-\varepsilon) + \varepsilon\right)\cos x}{\varepsilon + e^{\frac{1}{\varepsilon}}(\sin 1 - \varepsilon\cos 1)} \\ &+ \frac{\varepsilon e^{\frac{1}{\varepsilon} - \frac{x}{\varepsilon}}(\varepsilon(1-\cos 1) - 1 + \sin 1)}{\varepsilon + e^{\frac{1}{\varepsilon}}(\sin 1 - \varepsilon \cos 1)} + x - \varepsilon. \end{aligned}$$

The errors and convergence rates

$$\chi_N = \|u - u^h\|_{h,\varepsilon}, \qquad p_N = \frac{\ln \chi_N - \ln \chi_{2N}}{\ln 2}$$

on modified D-S and Shishkin meshes are presented in Table 2. The rate of convergence on mesh (3.1) is slightly larger than on the Shishkin mesh. It can be observed that numerical results on the D-S mesh outperform results on the Shishkin mesh in the sense that errors are smaller on mesh (3.1).

Table 3 shows uniform convergence of previously described method on the D-S mesh – the number of mesh points is fixed ($N = 512$) while parameter ε takes various values.

	modified D-S mesh		Shishkin mesh	
N	χ_N	p_N	χ_N	p_N
2^5	6.73e-02	0.872	6.70e-02	0.834
2^6	3.68e-02	0.911	3.76e-02	0.860
2^7	1.96e-02	0.931	2.07e-02	0.878
2^8	1.03e-02	0.941	1.13e-02	0.890
2^9	5.35e-03	0.946	6.08e-03	0.900
2^{10}	2.78e-03	0.949	3.26e-03	0.907
2^{11}	1.44e-03	0.951	1.74e-03	0.912
2^{12}	7.44e-04	–	9.23e-04	–

Table 2: Comparison of results on D-S and Shishkin meshes for $\varepsilon = 10^{-6}$.

ε	χ_N
10^{-2}	2.23e-03
10^{-3}	2.70e-03
10^{-4}	2.77e-03
10^{-5}	2.78e-03
10^{-6}	2.78e-03
10^{-7}	2.78e-03
10^{-8}	2.78e-03

Table 3: Uniform convergence for $N = 512$.

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