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BICAPACITIES ON BOUNDED LATTICES – BASIC PROPERTIES

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Abstract. The paper is a preliminary announcement of studying bipolar capacities on bounded lattices. An algebraic structure will be constructed where it is possible to define a bipolar capacity and one special case will be shown when the bipolar capacity is additive.

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1. Introduction

Bipolarity in solving decision-making problems has been used already for about 20 years. The theory of bipolar capacities and integral with respect to such capacities was introduced by Grabisch et al. [6, 7]. In some cases, a single value is not sufficient as a result. Interval-valued fuzzy sets or Atanasov's intuitionistic fuzzy sets (see [1]) are broadly used in fuzzy decision-making. Both, interval-valued fuzzy sets as well as Atanasov's intuitionistic fuzzy sets, are special cases of lattice-valued fuzzy sets, introduced by Goguen [5]. This paper contains some preliminary considerations in examining possible lattice-valued fuzzy sets where it is possible to define bipolar capacities. Readers are assumed to be familiar with basics of lattices. For details on the lattice theory readers are referred to the monograph [2]. The paper is organized as follows. Section 2 is devoted to recalling known results and notions that will be needed in authors' considerations. In Section 3 the main results will be formulated. Conclusions will be formulated in Section 4.

2. Preliminaries

In this section basic notions and results on bipolarity and also some types of lattices will be provided.

An important notion will be that of a complemented lattice.

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Definition 2.1 ([2])**.** Let $(L, \vee, \wedge, 0, 1, ^c)$ be a bounded lattice and $\cdot^c$ a decreasing function such that for every $x \in L$ there exists a uniquely given element x^c with

$$x \wedge x^c = 0, \qquad x \vee x^c = 1. \tag{2.1}$$

Then the lattice L is said to be *complemented* and the element x^c the *complement of* x.

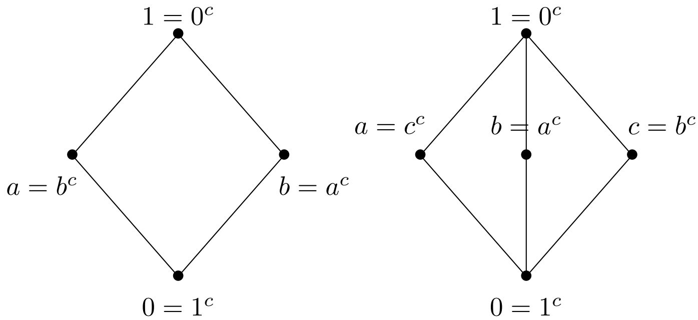

Figure 1: Examples of complemented lattices, left the lattice L_1, right L_2

Remark 2.2. Both of the lattice in Figure 1 are complemented, however, there is substantial difference between them. The lattice left has a so-called *involutive* complement, i.e., $(x^c)^c = x$ for all elements of L_1, while the complement in the lattice L_2 is not involutive.

Definition 2.3. Let X be a non-empty finite set and $\mathcal{P}(X)$ its powerset. A monotone set-function $\mu : \mathcal{P}(X) \to [0, 1]$ is said to be a *capacity* if $\mu(\emptyset) = 0$, $\mu(X) = 1$.

One of the main notions in this paper is that of a bicapacity. Its definition on a finite set X follows.

Definition 2.4 ([6])**.** Let X be a non-empty finite set and $\mathcal{P}(X)$ its powerset. Denote $\mathcal{Q}(X) = \{(A, B) \in \mathcal{P}(X) \times \mathcal{P}(X); A \cap B = \emptyset\}$. A function $h : \mathcal{Q}(X) \to [-1, 1]$ such that

1. h is increasing in the first variable,
2. h is decreasing in the second variable,
3. $h(\emptyset, \emptyset) = 0$, $h(X, \emptyset) = 1$, $h(\emptyset, X) = -1$,

is called a *bicapacity*.

Example 2.5. A typical example of a bicapacity on a non-empty finite set X is the following

$$h(A, B) = \mu(A) - \nu(B) \qquad \text{where } \mu \text{ and } \nu \text{ are capacities.} \tag{2.2}$$

Particularly, if $\text{card}(X) = n$, one may have

$$h_{\text{sym}}(A, B) = \frac{\text{card}(A)}{n} - \frac{\text{card}(B)}{n}, \tag{2.3}$$

where $(A, B) \in \mathcal{Q}(X)$ from Definition 2.4.

The bicapacity h_{sym} defined by formula (2.3), is called *symmetric* (see, e.g., [8, 9]).

An important notion will be also that of MV-algebra [4].

Definition 2.6 ([3, 4])**.** An *MV-algebra* is an algebra $(A, \oplus, \neg, 0)$ of type $(2, 1, 0)$, satisfying

(M1) $x \oplus y = y \oplus x$,

(M2) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(M3) $x \oplus 0 = x$,

(M4) $\neg\neg x = x$,

(M5) $x \oplus 1 = 1$ where $1 = \neg 0$,

(M6) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Remark 2.7. On any MV-algebra M, an order $\leq$ can be introduced in the following way ([3]):

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1. \tag{2.4}$$

Moreover, the order $(M, \leq)$ can be organized into a bounded distributive lattice $(M, \vee, \wedge, 0, 1)$ by

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg x \vee \neg y). \tag{2.5}$$

The operation $\neg$ is not a complement in the sense of Definition 2.1.

3. Main results

Complemented lattices and MV-algebras are main structures that motivated this research. The lattice L_2 depicted in Fig. 1 right, is a complemented lattices, however, some technical problems might occur when constructing a bicapacity on that lattice because the complement is not involutive. However, it is possible to define an algebraic structure on the lattice L_2 skipping the lattice-theoretical complement.

Example 3.1. Consider the lattice L_2 from Fig. 1 right skipping the complement. Define a partial binary operation $\oplus$ and a unary operation $\neg$ such that $0 \oplus x = x \oplus 0 = x$ and $1 \oplus x = x \oplus 1 = 1$ for all $x \in L_2$ and $\neg 0 = 1$, $\neg 1 = 0$, and results for the set of inputs $\{a, b, c\}$ is given by Table 1. The algebraic structure $(L_2, \oplus, \neg, 0)$ is not an MV-algebra and neither $(L_2, \vee, \wedge, 1, 0)$ is a complemented lattice with $\neg$ as the complement.

$\oplus$	a	b	c
a	a	$-$	1
b	$-$	1	$-$
c	1	$-$	c

$\neg$	a	b	c
	c	b	a

Table 1: Operation $\oplus$ (left) and $\neg$ (right) on the set $\{a, b, c\}$

It is possible to define a dual operation to $\oplus$, $\odot$, defined for any pair $(x, y) \in L_2 \times L_2$ whenever $\oplus$ is defined for (x, y), by

$$x \odot y = \neg(\neg x \oplus \neg y). \tag{3.1}$$

Then the following holds for $\neg$:

$$x \oplus \neg x = 1, \quad x \odot \neg x = 0. \tag{3.2}$$

This means that formula (2.1) a lattice-theoretical complement is 'mimicked' by $\neg$ (cf. formula (3.2)) in the corresponding algebraic structure, just with respect to $\oplus$ and $\odot$.

Remark 3.2. From now on, mentioning an algebraic structure $(A, \oplus, \neg)$ it will be assumed that formula (3.2) is fulfilled, where $\odot$ is given by formula (3.1).

Definition 3.3. Let $(A, \oplus, \neg)$ be an algebraic structure. Then $\overline{(A, \oplus, \neg)}$ will denote the *dual algebraic structure*, i.e., the algebraic structure with the reverted order.

When no confusion may occur, A will denote also the algebraic structure itself and $\overline{A}$ its dual algebraic structure. By $\overline{x}$ an element of $\overline{A}$ will be denoted.

Definition 3.4. Let $(A, \oplus, \neg, 0)$ be an algebraic structure. A monotone function $\mu : A \to [0, 1]$ is said to be a capacity if $\mu(0) = 0$, $\mu(1) = 1$, where $1 = \neg\, 0$.

Definition 3.5. Denote $\mathcal{Q}(A) = \{(C_1, C_2) \in A \times A; C_2 \le \neg C_1)\}$ for an algebraic structure A. A function $\mathcal{H} : \mathcal{Q}(A) \to A \cup \bar{A}$ will be called a *bipolar capacity* if the following properties are fulfilled

(B1) $\mathcal{H}$ is increasing in the first variable,

(B2) $\mathcal{H}$ is decreasing in the second variable,

(B3) $\mathcal{H}((1, 0) = 1$, $\mathcal{H}(0, 1) = \bar{1}$, $\mathcal{H}(0, 0) = 0$.

For a construction of a bipolar capacity on A, two capacities, $\mu : A \to [0, 1]$ and $\nu : A \to [0, 1]$ can be used.

Example 3.6. Let A be an algebraic structure and μ and ν two capacities on A. The following formula defines a bipolar capacity

$$\mathcal{H}(C_1, C_2) = \begin{cases} C_1 & \text{if } \mu(C_1) > \nu(C_2), \\ \overline{C_2} & \text{if } \nu(C_2) > \mu(C_1), \\ 0 & \text{if } \mu(C_1) = \nu(C_2). \end{cases}$$

Definition 3.7. A bipolar capacity $\overline{\mathcal{H}}$ is *dual* to $\mathcal{H}$ if the following holds for all $(C_1, C_2) \in \mathcal{Q}(A)$

$$\overline{\mathcal{H}}(C_2, C_1) = \mathcal{H}(C_1, C_2). \tag{3.3}$$

$\overline{\mathcal{H}}$ is said to be *self-dual* to $\mathcal{H}$ if $\overline{\mathcal{H}} = \mathcal{H}$.

Assume a bipolar capacity $\mathcal{H}$ is used in solving a decision-making problem. Then self-duality of $\mathcal{H}$ means that the preference for positive neither for negative evaluations is used. Particularly, the bipolar capacity $\mathcal{H}$ in Example 3.6 is self-dual if $\mu = \nu$.

Definition 3.8. Let A be an algebraic structure containing finitely many elements. A is said to be *atomic* if every element $x \in A$ can be decomposed as follows

$$x = \bigoplus_{i=1}^{k} a_i, \tag{3.4}$$

where a_i are elements such that $b \leq a_i$ implies $b = 0$ or $b = a_i$, Elements a_i are said to be *atoms.*

The expression of an element x as the sum of atoms is not necessarily unique.

Theorem 3.9. *Let A be an atomic algebraic structure. Let all atoms be enumerated by numbers from $N = \{1, 2, \ldots, n\}$. Assume that for all $x \in A$, if*

$$x = \bigoplus_{i-1}^{k} a_i = \bigoplus_{j=1}^{m} b_j \tag{3.5}$$

are two different decompositions into sets of atoms, then $k = m$. Then

$$\mathcal{H}(C_1, C_2) = \begin{cases} \bigoplus_{i=1}^{n_1 - n_2} a_i & \textit{if } n_1 > n_2, \\ \bigoplus_{i=1}^{n_2 - n_1} \overline{a_i} & \textit{if } n_1 < n_2, \\ 0 & \textit{if } n_1 = n_2, \end{cases} \tag{3.6}$$

is a bipolar capacity, where n_1 and n_2 are numbers of atoms of decompositions of C_1 and C_2, respectively. The atoms a_i, $\overline{a_i}$ are chosen in such a way that always atoms with smaller enumeration numbers are taken. Moreover, the bipolar capacity $\mathcal{H}$ is additive with respect to $\oplus$.

4. Conclusions

The paper is an announcement of preliminary results of bipolar capacities on lattices (and algebraic structures). The bipolar capacities have been defined and a special case that led to additive bipolar capacities, was shown in Theorem 3.9.

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