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THE ORNESS MEASURE FOR OWA AND BIOWA OPERATORS [1]

Martin Kalina [2], Biljana Mihailović [3] and Mirjana Štrboja [4]

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Abstract. The OWA operators are useful tools in muticriteria decision-making. The BIOWA operators are a generalization of OWA operators. The orness measure for some operator gives information about the similarity of aggregation to OR (Max) operator. In this paper an overview on OWA and BIOWA operator and their orness measure are presented.

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1. Introduction

An aggregation function is a function that processes together several numerical values to obtain a single representative value [2]. These functions have numerous applications, in economy, finance, computer sciences, image processing, etc. The ordered weighted averages (OWA) introduced by Yager [13] form a special class of aggregation functions that includes the arithmetic average, minimum, maximum and median. The OWA operators are useful in muticriteria decision-making, and also in other fields where ranking is important ([15]). As a generalization of OWA operators, the bipolar ordered weighted averages (BIOWA), recently were introduced by Stupňanová and Jin [12] and later studied by Mesiar et al. in [9]. These new bipolar aggregation functions are based on the bipolar Choquet integral [1]. Besides the bipolar Choquet integral, other types of bipolar integrals were introduced [3, 4, 10, 11]. The

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[2]Department of Mathematics, Faculty of Civil Engineering, Slovak University of Technology, Slovakia, e-mail: martin.kalina@stuba.sk

[3]Department of Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, e-mail: lica@uns.ac.rs

[4]Department of Mathematics and Informatics, Faculty of Sciences, University of Novi Sad, e-mail: mirjana.strboja@dmi.uns.ac.rs

notion of bipolarity in the framework of multi-criteria decision making was observed by Jin et al. in [5]. Recently, several other generalizations of OWA were studied ([6, 7, 8]).

The paper is organized as follows. In Section 2, we give an overview on the function aggregations and notations that is used in the rest of paper. In Section 3, we present the OWA operator with related the orness measure. The BIOWA operators and the measure of orness associated with BIOWA are given in Section 4.

2. Preliminaries

Let $\mathbb{I}$ be a nonempty subinterval of $[-\infty,\infty]$. A function of n variable $F:\mathbb{I}^n\to\mathbb{I}$ that is nondecreasing in each variable and that satisfies the boundary conditions

$$\inf_{(f_1,f_2,...,f_n)\in\mathbb{I}^n} F(f_1,f_2,...,f_n)=\inf\mathbb{I} \text{ i } \sup_{(f_1,f_2,...,f_n)\in\mathbb{I}^n} F(f_1,f_2,...,f_n)=\sup\mathbb{I}.$$

is an aggregation function on $\mathbb{I}$ [2]. Moreover, F is a symmetric aggregation function if

$$F(f_1,f_2,...,f_n)=F\left(f_{\tau(1)},f_{\tau(2)},...,f_{\tau(n)}\right),$$

for all $(f_1,f_2,...,f_n)\in\mathbb{I}^n$ and any permutation of indexes τ.

In the rest of this paper we consider a non-empty finite set $X=\{x_1,x_2,\ldots,x_n\}$ with $\mathrm{card}(X)=n$. Let f is a real-valued function on X, we denote $f(x_i)=f_i$, $i=1,\ldots,n$.

3. OWA operator

Let us recall the definition of an OWA operator proposed by Yager [13].

Definition 3.1. *An* OWA *operator is a mapping* $\mathrm{OWA}_{\mathbf{w}}:\mathbb{R}^n\to\mathbb{R}$ *that has an associated weighting vector* $\mathbf{w}=(w_1,\ldots,w_n)$ *with the following properties*

$$w_1+w_2+\cdots+w_n=1 \quad \text{and} \quad 0\le w_i\le 1,\ i=1,\ldots,n$$

given by

$$\mathrm{OWA}_{\mathbf{w}}(f)=\sum_{i=1}^{n} w_i f_{\alpha(n-i+1)},$$

where $\alpha=(\alpha(1),\ldots\alpha(n))$ *is a permutation of indexes such that* $f_{\alpha(1)}\le f_{\alpha(2)}\le\cdots\le f_{\alpha(n)}$*, for all* i*.*

In the next example some notable OWA operators are given.

Example 3.2. *For*

(i) $\mathbf{w}=\mathbf{w}^*$ *where* $\mathbf{w}^*=(1,\ldots,0)$ *we have* $\mathrm{OWA}_{\mathbf{w}^*}(f)=\max\{f_i\mid x_i\in X\}$,

(ii) $\mathbf{w}=\mathbf{w}_*$ *where* $\mathbf{w}_*=(0,\ldots,1)$ *we have* $\mathrm{OWA}_{\mathbf{w}_*}(f)=\min\{f_i\mid x_i\in X\}$,

(iii) $\mathbf{w}=\mathbf{w}_{Ave}$ *where* $\mathbf{w}_{Ave}=\left(\frac{1}{n},\ldots,\frac{1}{n}\right)$ *we have* $\mathrm{OWA}_{\mathbf{w}_{Ave}}(f)=\frac{1}{n}\sum_{i=1}^{n} f_i$.

It holds

$$\mathrm{OWA}_{\mathbf{w}_*}(f) \le \mathrm{OWA}_{\mathbf{w}}(f) \le \mathrm{OWA}_{\mathbf{w}^*}(f),$$

where $\mathbf{w}$ is an arbitrary weighting vector.

3.1. The orness measure for OWA operator

The orness measure for OWA operator was introduced by Yager [13].

Definition 3.3. *The orness measure for an* OWA *operator that has an associated weighting vector* $\mathbf{w}$ *is defined by*

$$\mathrm{orness}(\mathrm{OWA}_{\mathbf{w}}) = \frac{1}{n-1}\sum_{i=1}^{n}(n-i)\,w_i.$$

Obviously, we have

$$\mathrm{orness}(\mathrm{OWA}_{\mathbf{w}^*}) = 1,\ \mathrm{orness}(\mathrm{OWA}_{\mathbf{w}_*}) = 0 \text{ and } \mathrm{orness}(\mathrm{OWA}_{\mathbf{w}_{Ave}}) = 0.5.$$

The orness measure gives information about closeness some OWA operator to the operator $\mathrm{OWA}_{\mathbf{w}^*}$. If $\mathrm{orness}(\mathrm{OWA}_{\mathbf{w}}) \le 0.5$, then this operator is "orlike" operator, in the case that $\mathrm{orness}(\mathrm{OWA}_{\mathbf{w}}) \ge 0.5$ then it is "andlike" operator [14].

The measure of orness associated with a regular increasing quantifier (RIM), i.e., an increasing function $Q:[0,1]\to[0,1]$ that fulfills $Q(0)=0$ and $Q(1)=1$, also was observed in [14]. Hence, for $w_i = Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)$ we have the operator in the following form

$$\mathrm{OWA}_{\mathbf{w}^Q}(f) = \sum_{i=1}^{n}\left(Q\left(\frac{i}{n}\right) - Q\left(\frac{i-1}{n}\right)\right) f_{\alpha(n-i+1)}$$

and

$$\mathrm{orness}(\mathrm{OWA}_{\mathbf{w}^Q}) = \frac{1}{n-1}\sum_{i=1}^{n-1} Q\left(\frac{i}{n}\right).$$

The orness of a RIM quantifier Q is defined by

$$\mathrm{orness}(Q) = \lim_{n\to\infty}\frac{1}{n-1}\sum_{i=1}^{n-1} Q\left(\frac{i}{n}\right) = \int_0^1 Q(x)\,dx.$$

Example 3.4. *For*

(i) $Q(x) = Q^*(x)$ *where* $Q^*(x) = \begin{cases} 0, & x=0 \\ 1, & x>0 \end{cases}$ *we get weighting vector* $\mathbf{w}^*$ *and* $\mathrm{orness}(Q^*) = 1$,

(ii) $Q(x) = Q_*(x)$ *where* $Q_*(x) = \begin{cases} 0, & x<0 \\ 1, & x=0 \end{cases}$ *we get weighting vector* $\mathbf{w}_*$ *and* $\mathrm{orness}(Q_*) = 0$,

(iii) $Q(x) = Q_{Ave}(x)$ *where* $Q_{Ave}(x) = x,\ x\in[0,1]$, *we get weighting vector* $\mathbf{w}_{Ave}$ *and* $\mathrm{orness}(Q_{Ave}) = 0.5$.

If $Q(x) \ge x$ for all x, then Q is "orlike" and $\mathrm{orness}(Q) \ge 0.5$, and if $Q(x) \le x$ for all x, then Q is "andlike" and $\mathrm{orness}(Q) \le 0.5$ [14].

4. BIOWA operator

In this section BIOWA operator and related the orness measure are presented, according to [9]. Denote

$$\widetilde{N} = \{(i,j) \mid i,j \in \{0,1,\dots,\mathrm{card}(X)\}, i+j \le \mathrm{card}(X)\}.$$

Let $t : \widetilde{N} \to [-1,1]$ be a function such that the following conditions are satisfied:

(i) $t(n,0) = 1$, $t(0,0) = 0$, $t(0,n) = -1$,

(ii) t is increasing in the first coordinate,

(iii) t is decreasing in the second coordinate.

The class of functions $f : X \to \mathbb{R}$ is denoted by $\mathcal{S}$. For $f \in \mathcal{S}$, we use notations

$$X^{+0} = \{x_i \in X \mid f(x_i) \ge 0\}, \ X^{-} = \{x_i \in X \mid f(x_i) < 0\}.$$

Observe a function $s : X \to \{(1,0),(0,1)\}$ defined by

$$s(x_i) = \begin{cases} (1,0), \ x_{\alpha(i)} \in X^{+0}, \\ (0,1), \ x_{\alpha(i)} \in X^{-}, \end{cases} \tag{4.1}$$

where α is any permutation of indexes such that $|f_{\alpha(1)}| \le |f_{\alpha(2)}| \le \cdots \le |f_{\alpha(n)}|$. Denote

$$w_i^s = t\left(\sum_{j=i}^{n} s(x_j)\right) - t\left(\sum_{j=i+1}^{n} s(x_j)\right), \tag{4.2}$$

where $\sum_{j=n+1}^{n} s(x_j) = (0,0)$.

An BIOWA operator $\mathrm{BIOWA}_t : \mathbb{R}^n \to \mathbb{R}$ is given by

$$\mathrm{BIOWA}_t(f) = \sum_{i=1}^{n} w_i^s \cdot |f_{\alpha(i)}|,$$

where weights w_i^s and s are given by (4.1) and (4.2), respectively. The vector $\mathbf{w}_s = (w_1^s, .., w_n^s)$ is weighting vector, w_i^s are weights and function t is called a generating function. Any BIOWA operator is a symmetric aggregation function on $\mathbb{R}$.

Example 4.1. *For*

(i) $t(x,y) = t^*(x,y)$ *where* $t^*(x,y) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, y < n, \\ -1, & y = n, \end{cases}$ *we have* $\mathrm{BIOWA}_{t^*}(f) = \max\{f_i \mid x_i \in X\}$,

(ii) $t(x,y) = t_*(x,y)$ *where* $t_*(x,y) = \begin{cases} 1, & x = n, \\ 0, & x < n, y = 0, \\ -1, & y < n, \end{cases}$ *we have* $\mathrm{BIOWA}_{t_*}(f) = \min\{f_i \mid x_i \in X\}$,

(iii) $t(x,y) = t_{Ave}(x,y)$ *where* $t_{Ave}(x,y) = \frac{x-y}{n}$, *we have* $\mathrm{BIOWA}_{t_{Ave}}(f) = \frac{1}{n}\sum_{i=1}^{n} f_i$.

The BIOWA operator $\mathrm{BIOWA}_t : \mathbb{R}^n \to \mathbb{R}$ is OWA operator if and only if $t(x,y) = t_g(x,y) = g(x) - 1 + g(n-y)$, where $g : \{0,1,\dots,n\} \to [0,1]$ is an increasing function that fulfills the conditions $g(0) = 0$ and $g(n) = 1$. In that case $w_i = g(n-i+1) - g(n-i)$ and

$$\mathrm{BIOWA}_{t_g}(f) = \sum_{i=1}^{n} w_i \cdot f_{\gamma(i)},$$

where γ is a permutation of indexes such that $f_{\gamma(1)} \le f_{\gamma(2)} \le \cdots \le f_{\gamma(n)}$.

4.1. The orness measure for BIOWA operator

The orness measure for BIOWA operator was introduced in [9].

Definition 4.2. *The orness measure for an* BIOWA *operator* BIOWA_t *is defined by*

$$\mathrm{orness}(\mathrm{BIOWA}_t) = \frac{1}{(n-1)(n+2)} \sum_{(i,j)\in\widetilde{N}} t(i,j) + 0.5.$$

It holds:

$$\mathrm{orness}(\mathrm{BIOWA}_{t^*}) = 1,\ \mathrm{orness}(\mathrm{BIOWA}_{t_*}) = 0 \text{ and } \mathrm{orness}(\mathrm{BIOWA}_{t_{Ave}}) = 0.5.$$

In [9] the BIOWA-quantifiers were introduced and similar as the orness for RIM quantifiers, the orness for BIOWA-quantifiers was proposed.

Let us observe the lattice $(L, \le_L)$ where $L = \left\{(u,v) \in [0,1]^2 \mid u + v \le 1\right\}$ and $(u_1, v_1) \le_L (u_2, v_2)$ iff $u_1 \le u_2$ and $v_1 \le v_2$. If $T : L \to [-1,1]$ is an order homomorphism between lattices $(L, \le_L)$ and $([0,1], \le)$ satisfying $T(0,0) = 0$, then T is called a BIOWA-quantifiers.

The orness of a BIOWA-quantifier T is defined by

$$\mathrm{orness}(T) = \iint_L T(u,v)\, du dv + 0.5.$$

Example 4.3. *For*

(i) $T(u,v) = T^*(u,v)$ *where* $T^*(u,v) = \begin{cases} 1, & u > 0, \\ 0, & u = 0, v < 1, \\ -1, & v = 1, \end{cases}$ *we have* $\mathrm{orness}(T^*) = 1$,

(ii) $T(u,v) = T_*(u,v)$ *where* $T_*(u,v) = \begin{cases} 1, & u = 1, \\ 0, & u < 1, v = 0, \\ -1, & v < 1, \end{cases}$ *we have* $\mathrm{orness}(T_*) = 0$,

(iii) $T(u,v) = T_{Ave}(u,v)$ *where* $T_{Ave}(u,v) = u - v$, *we have* $\mathrm{orness}(T_{Ave}) = 0.5$.

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