The 9th Conference on Mathematics in Engineering: Theory and Applications
Novi Sad, May 31st–June 2nd 2024

”EQUATION SOLVING” GENERALIZED INVERSES - WHAT ARE THEY?[1]

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https://doi.org/10.24867/META.2024.11 Short communication

Abstract. A new block representation theorem for the core inverse of square matrix with the index less or equal to 1 is presented. Two examples are given and some open problems are stated, having in mind an application of generalized inverses in solving fuzzy linear systems.

AMS Mathematics Subject Classification (2020): 15A06, 15A09, 15A10

Key words and phrases: system of linear equations, {1}-inverse, Drazin inverse, core inverse, BT inverse

1. Introduction

The most prominent generalized inverse of finite matrices (square or rectangle) in literature known as the Moore-Penrose inverse or ”pseudo-inverse”, was originally introduced by E.H. Moore (1920) and rediscovered by R. Penrose (1955). Moore’s first publication on this unique inverse of matrices, its main properties and its application to linear equations, appeared in 1920, as an abstract of a talk given at a meeting of the American Mathematical Society. Later, his detailed results on the subject were published in 1935. However, some authors considered that Moore’s discovery has been obtained much earlier, probably in 1906. Since only very dedicated readers could understand unnecessarily complicated Moore’s notation, his work was sinking into oblivion. Penrose rediscovered the general reciprocal, nowadays called the Moore-Penrose inverse, 35 years after Moore’s first publication. In 1951, Bjerhammar, who first rediscovered Moore’s inverse, also noted the relationship of generalized inverses to solutions of linear systems, whereas, in 1955 Penrose sharpened and extended Bjerhammar’s results on linear systems, and showed that Moore’s inverse, for a given matrix F, is the unique matrix G satisfying the four equations, nowadays called Penrose’s equations. Since 1955 thousands of papers on various aspects of generalized inverses and their applications have appeared.

For a given matrix F, generalized inverses satisfying the first Penrose’s equation, e.g. $FGF = F$, are called {1}-inverses or inner inverses. Recently, in [1], Baksalary and Trenkler introduced the core inverse of square matrices with index less or equal to 1, whereas, in [2] they proposed BT inverses. The application of generalized inverses in solving fuzzy linear systems of Friedman et al.’s type was consequently studied by

[1]The author acknowledges the financial support of Department of Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, the project ”Improving the teaching process in the English language in fundamental disciplines”.

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numerous authors. In particular, recently, B. Mihailović et al. [7] showed the general algebraic form of fuzzy linear systems based on an $\{1\}$-inverse of its coefficient matrix. Very recently, the first straightforward method for solving dual fuzzy linear systems using the block structure of an arbitrary $\{1\}$-inverse was introduced by Dragić et al. in [4]. In [6], the core inverse was considered for solving fuzzy linear systems, whereas in [5] the method for solving fuzzy linear systems using the block structure of BT inverse was introduced.

The paper is organized as follows. In Section 1, some preliminaries related to generalized inverses are presented. In Section 2, a new block structure of the core inverse is presented and two open problems are stated.

2. Generalized inverses

Throughout this paper, we denote the set of all $m \times n$ real matrices by $\mathcal{M}^{m\times n}$. For $F \in \mathcal{M}^{m\times n}$ the symbols F^T, F^{-1}, $rank(F)$, $\mathcal{N}(F)$, and $\mathcal{R}(F)$ will stand for the transpose, the ordinary inverse ($m = n$), the rank, the kernel, and the range space of F, respectively. Moreover, let us denote with $\mathcal{M}^n$ the class of all square $n \times n$ real matrices, I_n denotes the identity matrix of order n, and O denotes the null matrix of order n. For $F \in \mathcal{M}^n$, the index of F, denoted by $Ind(F)$, is the smallest non-negative integer k such that $rank(F^{k+1}) = rank(F^k)$. For each $F \in \mathcal{M}^n$, define $F^0 = I_n$. For a non-singular matrix F it holds $Ind(F) = 0$, since $rank(F) = rank(F^0) = rank(I_n) = n$, while $Ind(O) = 1$, since $rank(O^2) = rank(O) = 0$ and $rank(O^0) = n$.

First, let us recall the system of four Penrose's equations for $F \in \mathcal{M}^n$:

$$\text{(P1)} \qquad FGF = F,$$

$$\text{(P2)} \qquad GFG = G,$$

$$\text{(P3)} \qquad (FG)^{\mathrm{T}} = FG,$$

$$\text{(P4)} \qquad (GF)^{\mathrm{T}} = GF,$$

where matrix $G \in \mathcal{M}^{n\times m}$ is unknown [3]. Let us consider the additional matrix equations as follows:

$$\text{(P1')} \qquad FGF^2 = F^2,$$

$$\text{(P2')} \qquad FG^2 = G,$$

$$\text{(P4')} \qquad (F^TGF^2)^T = F^TGF^2,$$

$$\text{(P5)} \qquad FG = GF,$$

$$\text{(P5')} \qquad GF^{k+1} = F^k.$$

Definition 2.1. ([3]) For any $F \in \mathcal{M}^{m\times n}$, let $\mathcal{H}\{i, j, \dots h\}$ denote the set of matrices $G \in \mathcal{M}^{n\times m}$ which fulfill equations $(Pi), (Pj), \dots, (Ph)$ among the equations (P1) to (P5'). A matrix $G \in \mathcal{H}\{i, j, \dots, h\}$ is called an $\{i, j, \dots, h\}$-inverse of F and it will be denoted by $F^{(i,j,\dots,h)}$.

According to [1, 2, 6], let us present the most prominent generalized inverse, the Moore-Penrose inverse, and some recently introduced inverses, the BT inverse and the core inverse.

Definition 2.2. Let $F \in \mathcal{M}^{m\times n}$.

(i) Let $G \in \mathcal{M}^{n\times m}$ be a matrix which fulfills the system of four matrix equations (P1)-(P4). This matrix G is called *the Moore-Penrose inverse* of F, and it is denoted by $F^\dagger$ or by $F^{(1,2,3,4)}$.

(ii) Let $F \in \mathcal{M}^n$ with $Ind(F) = k$, and let $G \in \mathcal{M}^n$ be a matrix which fulfills the system of three matrix equations (P2), (P5) and (P5'). This matrix G is called *the Drazin inverse* of F, and denoted by F^D or by $F^{(2,5,5')}$. Moreover, if $Ind(F) \leq 1$, the Drazin inverse G is called *the group inverse* of F, and denoted by $F^{\#}$ or by $F^{(1,2,5)}$.

(iii) Let $F \in \mathcal{M}^n$ with $Ind(F) = k$, and let $G \in \mathcal{M}^n$ be a matrix which fulfills the system of four matrix equations (P1'), (P2), (P3) and (P4'). This matrix G is called *the BT inverse* of F, and it is denoted by $F^{\diamond}$ or by $F^{(1',2,3,4')}$.

(iv) Let $F \in \mathcal{M}^n$ with $Ind(F) \leq 1$, and let $G \in \mathcal{M}^n$ be a matrix which fulfills the system of three matrix equations (P1), (P2') and (P3). This matrix G is called *the core inverse* of F, and denoted by $F^{\textcircled{\#}}$ or by $F^{(1,2',3)}$.

Trivially, for each non-singular square matrix all those generalized inverses are equal to its ordinary inverse. The group inverse exists for all $F \in \mathcal{M}^n$, such that $Ind(F) \leq 1$ and it holds $F^{\#} = F^D$, i.e., in that case the group inverse of a matrix F is identical to its Drazin inverse. Obviously, (P1) implies (P1'), (P4) implies (P4'), but not vice versa. It holds $F^{\diamond} = (F^2F^{\dagger})^{\dagger}$ and if F is a singular matrix, with $Ind(F) = 1$, it holds $F^{\diamond} = F^{\textcircled{\#}} = F^{\#}FF^{\dagger}$. Since the Moore-Penrose inverse and the group inverse are uniquely determined, the core inverse and the BT inverse of $F \in \mathcal{M}^n$ are uniquely determined. Recall, a matrix $F \in \mathcal{M}^n$ that satisfies $F^{\dagger}F = FF^{\dagger}$ is called an EP matrix.

For $F \in \mathcal{M}^n$, $F = [f_{ij}]$, we denote with $|F|$ the matrix whose entries are the absolute values of entries of F, i.e., $|F| = [|f_{ij}|]$, $|F| \in \mathcal{M}^n$. We say that F is non-negative matrix if $f_{ij} \geq 0$, for all i, j.

Example 2.3. Let us consider the next singular matrices:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, \qquad |A| = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

In Table 1, the group inverse, the Drazin inverse and the Moore-Penrose inverse, whereas in Table 2, the core inverse, the BT inverse and an arbitrary $\{1\}$-inverse of these matrices are obtained.

F	$F^{\#}$	F^D	$F^{\dagger}$
A, $Ind(A) = 2$	$--$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$
$\|A\|$, $Ind(\|A\|) = 1$	$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$

Table 1: Generalized inverses of A and $|A|$

Obviously, $|A|$ is a singular matrix with $Ind(|A|) = 1$, it is symmetric, therefore $|A|^{\#} = |A|^{\dagger}$, and it is an EP matrix. On the other hand, the matrix A is a nilpotent matrix satisfying $A^2 = O$, and its index equals 2. Let us solve the next linear system:

$$\begin{array}{rcrcl} x_1 & - & x_2 & = & 5 \\ x_1 & - & x_2 & = & 5 \end{array}.$$

F	$F^{\textcircled{\#}}$	$F^{\diamond}$	$F^{(1)}$
A, $Ind(A)=2$	$--$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1-a+b+c & a \\ b & c \end{bmatrix}$, $a,b,c \in \mathbb{R}$
$\|A\|$, $Ind(\|A\|)=1$	$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} 1-a-b-c & a \\ b & c \end{bmatrix}$, $a,b,c \in \mathbb{R}$

Table 2: Generalized inverses of A and $|A|$

As it is well-known, the general solution is given in the next form (see, [3]):

$$X = A^{(1)}B + (I_n - A^{(1)}A)V, \tag{2.1}$$

where $A^{(1)}$ denotes any of $\{1\}$-inverses of A and $V = (v_1, \ldots, v_n)^T$ is arbitrary. Using e.g. $A^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, we compute, $A^{(1)}B = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, where $B = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$, and therefore, $X = \begin{bmatrix} 5 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, for arbitrary $v_1, v_2 \in \mathbb{R}$. Finally, $X = \begin{bmatrix} 5+p \\ p \end{bmatrix}$, $p \in \mathbb{R}$, $SS = \{ (x_1, x_2) \,|\, x_1 = 5+p, x_2 = p, p \in \mathbb{R} \}$, where SS stands for the solution set of this system of linear equations. Notice that neither $A^{\diamond}B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a solution of this system, nor $A^D B$.

In general, the Drazin inverse and the BT inverse are not "equation solving" generalized inverses. In the case of square linear systems, when the index of the coefficient matrix is less or equal to 1, we can compute the group inverse and the core inverse that are both "equation solving" generalized inverses, i.e. they are $\{1\}$-inverses of A. However, consider the consistent system of linear equations $x_1 - x_2 = 5$, $x_1 - x_2 = 5$, $x_3 = 3$, with the coefficient matrix M, then $M^{\diamond}B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ and $(0,0,3)$ is not a solution, but according to Theorem 2.5 in [5], the system $x_1 - x_2 = 0$, $x_1 - x_2 = 0$, $x_3 = 3$, has a solution $M^{\diamond}B_2$ because $B_2 \in \mathcal{R}(M^2)$.

3. A block structure of generalized inverses

In this section, a new block representation theorem of the core inverse is presented. Also, some open problems related to nonnegativity and block structure of "equation solving" generalized inverses are stated.

First, recall that for any $F \in \mathcal{M}^{m \times n}$ of rank r there exist matrices $Q \in \mathcal{M}^{m \times m}$ and $P \in \mathcal{M}^{n \times n}$ such that:

$$QFP = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = E_r, \tag{3.1}$$

where $I_r \in \mathcal{M}^r$ is the identity matrix. A block representation of any $\{1\}$-inverse of F is presented in the next theorem from [7].

Theorem 3.1. *Let $F \in \mathcal{M}^{m\times n}$ be a matrix of rank r. Let Q and P be non-singular, square matrices which fulfill (3.1). A matrix $G \in \mathcal{M}^{n\times m}$ is a solution of the matrix equation $FGF = F$ if and only if*

$$G = P \cdot \begin{bmatrix} I_r & Z_1 \\ Z_2 & Z_3 \end{bmatrix} \cdot Q, \tag{3.2}$$

where $I_r \in \mathcal{M}^r$ is the identity matrix, and $Z_1 \in \mathcal{M}^{r\times(m-r)}$, $Z_2 \in \mathcal{M}^{(n-r)\times r}$ and $Z_3 \in \mathcal{M}^{(n-r)\times(m-r)}$ are arbitrarily chosen matrices.

We will present a new method for computing the core inverse. For each $F \in \mathcal{M}^n$, $rank(F) = r$, there exist non-singular matrices $P, Q \in \mathcal{M}^n$, such that (3.1) holds (for details, see [7]). Such matrices P and Q are not uniquely determined, and they can be obtained by making the same elementary operations by rows or by columns on a matrix F and the identity matrix $I_n \in \mathcal{M}^n$, which provides:

$$\begin{bmatrix} F & I_n \\ I_n & 0 \end{bmatrix} \sim \begin{bmatrix} E_r & Q \\ P & 0 \end{bmatrix}.$$

Further, the product of such matrices Q and P and the product $Q \cdot Q^T$ admit the next block structures:

$$Q \cdot Q^{\mathrm{T}} = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix} \quad \text{and} \quad Q \cdot P = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}, \tag{3.3}$$

where $V_4 \in \mathcal{M}^{n-r}$ denotes a square sub-matrix of order $n-r$, and appropriate sub-matrices are $W_1 \in \mathcal{M}^r$, $W_2 \in \mathcal{M}^{r\times(n-r)}$, $W_3 \in \mathcal{M}^{(n-r)\times r}$, $W_4 \in \mathcal{M}^{n-r}$.

A new representation of the core inverse is presented in the next theorem.

Theorem 3.2. *(The core inverse) Let $F \in \mathcal{M}^n$, $rank(F) = r$ and $Ind(F) = 1$. Let Q and P be non-singular, square matrices which fulfill (3.1), such that the products $Q \cdot P$ and $Q \cdot Q^T$ admit (3.3). A matrix $G \in \mathcal{M}^n$ is the unique solution of the system of three matrix equations (P1), (P2') and (P3) if and only if*

$$G = P \cdot \begin{bmatrix} I_r & -W_2 \cdot W_4^{-1} \\ -V_4^{-1} \cdot V_3 & V_4^{-1} \cdot V_3 \cdot W_2 \cdot W_4^{-1} \end{bmatrix} \cdot Q, \tag{3.4}$$

where $V_4 \in \mathcal{M}^{n-r}$ is non-singular.

The next example and some directions for further investigations related to block structure of generalized inverses of the block matrix S_A, for $A \in \mathcal{M}^{m\times n}$ conclude this paper. Denote $A^+ = \frac{1}{2}(A + |A|)$, $A^- = \frac{1}{2}(|A| - A)$.

Example 3.3. Let $|A|$ be matrix considered in Example 2.3. Then for

$$S_{|A|} = \begin{bmatrix} |A|^+ & |A|^- \\ |A|^- & |A|^+ \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad S_{|A|}^{\dagger} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \end{bmatrix},$$

it holds $S_{|A|}^{\dagger} = S_{|A|}^{D} = S_{|A|}^{\#} = S_{|A|}^{\textcircled{\#}} = S_{|A|}^{\diamond}$ (by Theorem 4([7]), Theorem 5([7]), Theorem 3.2 and Example 2.3, since $Ind(S_{|A|}) = 1$ and $|A|$ is EP matrix).

By Theorem 3 from [7], $G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}$, $G_1 = S_{|A|}^{(1)}$, is obtained by

$|A|^{(1)} = \begin{bmatrix} 1 & 0 \\ -2 & 2 \end{bmatrix}$. However, one of $\{1\}$-inverses of $S_{|A|}$ is also:

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; G_2 = S_{|A|}^{(1)}.$$

Open problem 1: A characterization of the class of matrices A such that each $\{1\}$-inverse of S_A admits the symmetric block structure.
Open problem 2: A characterization of the class of matrices A such that each $\{1\}$-inverse of S_A is non-negative.

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