The 9th Conference on Mathematics in Engineering: Theory and Applications
Novi Sad, May 31st - June 2nd 2024

BIPOLAR FUZZY LINEAR SYSTEMS WITH A UNIQUE SOLUTION[1]

Ljubo Nedović [2], Biljana Mihailović [3] and Đorđe Dragić [4]

https://doi.org/10.24867/META.2024.13 Short communication

Abstract. For a given bipolar fuzzy number vector Y and a real matrix A, the system of equations $AX = Y$ is called a bipolar fuzzy linear system, where X is an unknown bipolar fuzzy number vector. We present a new method for solving bipolar fuzzy linear systems with a unique solution and illustrate this new approach by an example.

AMS Mathematics Subject Classification (2020): 03E72, 15A09

Key words and phrases: bipolar fuzzy numbers, bipolar fuzzy linear systems, completely non-singular matrix

1. Introduction

Fuzzy linear systems (FLS), introduced by Friedman et al. in [5], arose as a generalization of linear systems. There are numerous papers devoted to FLS of Friedman et al.'s type. In [2] Allahviranloo and Ghanbari presented a new, efficient method for obtaining exact algebraic solutions of a square FLS whose coefficient matrix is non-singular. The application of inner inverses ({1}-inverses) in solving fuzzy linear systems was consequently studied by numerous authors. A general algebraic solution of fuzzy linear systems of Friedman et al.'s type was characterized for the first time by Mihailović et al. in [6]. Also, the algorithm for solving non-square FLS based on the Moore-Penrose inverse of its coefficient matrix was presented in this paper. Recently, in [8], a straightforward method for solving $m \times n$ FLS $A\tilde{X} = \tilde{Y}$ was introduced, as a generalization of the obtained results from [6, 7]. The first straightforward method for solving dual fuzzy linear systems using arbitrary {1}-inverses of its coefficient matrices was introduced by Dragić et al. in [3, 4].

Bipolar fuzzy linear systems (BFLS) were introduced in [1]. The main aim of this paper is to propose a new method for solving square BFLS with a unique

[1]The authors acknowledge the financial support of the Ministry of Science, Technological Development and Innovation of the Republic of Serbia, Serbian Slovak bilateral project: A Bipolar Approach in Mathematical Models of Decision-Making Processes.

[2]Department of Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, e-mail: nljubo@uns.ac.rs

[3]Department of Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, e-mail: lica@uns.ac.rs

[4]Department of Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, e-mail: djordje.dragic@uns.ac.rs

solution. The paper is organized as follows. In Section 1, some preliminaries related to bipolar fuzzy numbers and bipolar fuzzy linear systems are presented. In Section 2, the general form of unique (strong) solutions of square BFLS is presented and an illustrative example is given.

2. Bipolar fuzzy linear systems

Recall some basic definitions and terminology ([1, 5, 7]).

Definition 2.1. [1] A bipolar fuzzy set (BFS) u in parametric form is a quadruple $\prec \underline{u}^P, \overline{u}^P, \underline{u}^N, \overline{u}^N \succ$ of the functions $\underline{u}^P(r)$, $\overline{u}^P(r)$, $\underline{u}^N(s)$, $\overline{u}^N(s)$; $0 \leq r \leq 1$, $-1 \leq s \leq 0$, satisfying the next conditions:

(i) $\underline{u}^P(r)$ is a bounded monotonically increasing (non-decreasing) left-continuous function on a set $(0,1]$ and right-continuous at point 0,

(ii) $\overline{u}^P(r)$ is a bounded monotonically decreasing (non-increasing) left-continuous function on a set (0, 1] and right-continuous at point 0,

(iii) $\underline{u}^N(s)$ is a bounded monotonically decreasing (non-increasing) left-continuous function on a set $(-1,0]$ and right-continuous at point -1,

(iv) $\overline{u}^N(s)$ is a bounded monotonically increasing (non-decreasing) left-continuous function on a set $(-1,0]$ and right-continuous at point -1,

(v) $\underline{u}^P(r) \leq \overline{u}^P(r)$,

(vi) $\underline{u}^N(s) \leq \overline{u}^N(s)$.

The set of all bipolar fuzzy numbers will be denoted by $\mathcal{B}$. For $u, v \in \mathcal{B}$, in parametric form $u = \prec \underline{u}^P, \overline{u}^P, \underline{u}^N, \overline{u}^N \succ$, $v = \prec \underline{v}^P, \overline{v}^P, \underline{v}^N, \overline{v}^N \succ$ any real number k, we define:

1. *Addition:* $u + v = \prec \underline{u}^P + \underline{v}^P, \overline{u}^P + \overline{v}^P, \underline{u}^N + \underline{v}^N, \overline{u}^N + \overline{v}^N \succ$,

2. *Scalar multiplication:* $$ku = \begin{cases} \prec k\underline{u}^P, k\overline{u}^P, k\underline{u}^N, k\overline{u}^N \succ, & k \geq 0, \\ \prec k\overline{u}^P, k\underline{u}^P, k\overline{u}^N, k\underline{u}^N \succ, & k < 0, \end{cases}$$

3. *Equality:* $u = v \Leftrightarrow \underline{u}^P = \underline{v}^P,\ \overline{u}^P = \overline{v}^P,\ \underline{u}^N = \underline{v}^N,\ \overline{u}^N = \overline{v}^N$.

We will use the next notation and terminology. Let $\mathcal{BV}_n$ denotes the class of all n-dimensional bipolar fuzzy number vectors. Let $X = (x_1, \ldots, x_n)^T$ denotes a bipolar fuzzy number vector, where $x_i \in \mathcal{B}$, $x_i = \prec \underline{x}_i^P, \overline{x}_i^P, \underline{x}_i^N, \overline{x}_i^N \succ$, for all $i = 1, \ldots, n$. For $X \in \mathcal{BV}_n$, the associated $2n \times 1$ classical functional vectors $X^P = (\underline{x}_1^P, \ldots, \underline{x}_n^P, -\overline{x}_1^P, \ldots, -\overline{x}_n^P)^T$, $X^N = (\underline{x}_1^N, \ldots, \underline{x}_n^N, -\overline{x}_1^N, \ldots, -\overline{x}_n^N)^T$ such that all its components are the unit interval functions, will be called the representative vectors for X. For any $X \in \mathcal{BV}_n$, the associated $n \times 1$ classical functional vectors $\underline{X}^P$ and $\underline{X}^N$, (resp. $\overline{X}^P$ and $\overline{X}^N$) with the lower (resp. upper) branches as their components, are $\underline{X}^P = (\underline{x}_1^P, \ldots, \underline{x}_n^P)^T$ and $\underline{X}^N = (\underline{x}_1^N, \ldots, \underline{x}_n^N)^T$ (resp. $\overline{X}^P = (\overline{x}_1^P, \ldots, \overline{x}_n^P)^T$ and $\overline{X}^N = (\overline{x}_1^N, \ldots, \overline{x}_n^N)^T$). We will use the notation $\overline{X}^P \geq \underline{X}^P$ if and only if for all $i = 1, \ldots, n$, and for each $r \in [0,1]$, it holds $\overline{x}_i^P(r) \geq \underline{x}_i^P(r)$, and similarly $\overline{X}^N \geq \underline{X}^N$.

For a given $Y \in \mathcal{BV}_n$ and real square matrix A of order n, *the bipolar fuzzy linear system* in the matrix form is $AX = Y$, where $X \in \mathcal{BV}_n$ is unknown.

For each real square matrix A of order n, denote $A^+ = \frac{1}{2}(A + |A|)$, $A^- = \frac{1}{2}(|A| - A)$, where $|A|$ is a square real matrix of order n whose entries are the absolute values of entries of A. Let S_A be a square matrix of order $2n$ defined by

$$S_A = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}. \tag{2.1}$$

Definition 2.2. For a given $Y \in \mathcal{BV}_n$ and real square matrix A of order n, *a solution* of BFLS $AX = Y$ is any $U \in \mathcal{BV}_n$ such that for all $r \in [0,1]$ it holds $S_A U^P(r) = Y^P(r)$ and for all $s \in [-1,0]$ it holds $S_A U^N(s) = Y^N(s)$.

3. A unique solution of BFLS

As a new result, we present a necessary condition for the consistency of BFLS with completely non-singular coefficient matrix. According to [2, 5], recall that A is a completely non-singular matrix if both matrices A and $|A|$ are non-singular and S_A is non-singular if and only if A is completely non-singular.

Theorem 3.1. *Let $AX = Y$ be a BFLS, where A is a completely non-singular matrix of order n and $Y \in \mathcal{BV}_n$. Let $X^{*P} = S_M Y^P$ and $X^{*N} = S_M Y^N$, where $M = A^{-1}$ and S_M is given by (2.1). Let $\mathsf{R}^P = Y^P - S_A X^{*P}$ and $\mathsf{R}^N = Y^N - S_A X^{*N}$.*

If the bipolar fuzzy linear system $AX = Y$ is consistent, then there exist $\mathsf{V}^P = \begin{bmatrix} V^P \\ V^P \end{bmatrix}$, and $\mathsf{V}^N = \begin{bmatrix} V^N \\ V^N \end{bmatrix}$, where $V^P = (v_1^P(r), \ldots, v_n^P(r))^T$ and $V^N = (v_1^N(s), \ldots, v_n^N(s))^T$, such that for all $r \in [0,1]$ and $s \in [-1,0]$ it holds $S_A \mathsf{V}^P = \mathsf{R}^P$, $S_A \mathsf{V}^N = \mathsf{R}^N$, $2V^P \le \overline{X}^{*P} - \underline{X}^{*P}$, and $2V^N \le \overline{X}^{*N} - \underline{X}^{*N}$.

Moreover, a unique solution X of BFLS is determined by $X^P = X^{*P} + \mathsf{V}^P$ and $X^N = X^{*N} + \mathsf{V}^N$.

Example 3.2. Let us solve the next 2×2 bipolar fuzzy linear system:

$$\begin{array}{rcrcl} x_1 & + & x_2 & = & \prec -2+2r, 5-5r, -2-4s, 13+11s \succ \\ x_1 & - & 2x_2 & = & \prec -6+3r, 3-6r, -17-4s, 1+14s \succ \end{array}.$$

Matrices A, A^+ and A^- of this BFLS are:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \; A^+ = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \; A^- = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Since both matrices A and $|A|$ are invertible, the matrix A is completely non-singular. The classical inverse of A is:

$$M = A^{-1} = -\frac{1}{3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}.$$

First, we compute $X^{*P} = S_M Y^P$, and $X^{*N} = S_M Y^N$, and obtain:

$$\begin{aligned} X^{*P} &= \left(-\frac{10}{3} + \frac{7}{3}r, -\frac{5}{3} + \frac{8}{3}r, -\frac{13}{3} + \frac{16}{3}r, -\frac{11}{3} + \frac{8}{3}r\right)^T, \\ X^{*N} &= (-7-4s, -1-6s, -9-12s, -10-5s)^T. \end{aligned}$$

We obtain $\mathsf{R}^P = Y^P - S_A X^{*P} = \left(3-3r, \frac{14}{3} - \frac{14}{3}r, 3-3r, \frac{14}{3} - \frac{14}{3}r\right)^T$ and $\mathsf{R}^N = Y^N - S_A X^{*N} = (6+6s, 10+10s, 6+6s, 10+10s)^T$. Further, the unique solutions of $S_A \mathsf{V}^P = \mathsf{R}^P$ and $S_A \mathsf{V}^N = \mathsf{R}^N$ are:

$$\begin{aligned} \mathsf{V}^P &= \left(\frac{4}{3}(1-r), \frac{5}{3}(1-r), \frac{4}{3}(1-r), \frac{5}{3}(1-r)\right)^T, \\ \mathsf{V}^N &= (2(1+s), 4(1+s), 2(1+s), 4(1+s))^T, \end{aligned}$$

respectively. In order to obtain a bipolar fuzzy number vector, i.e., in order to $\overline{x}_i^P(r) \geq \underline{x}_i^P(r)$ be fulfilled for $i = 1, 2$ and each $r \in [0,1]$, V^P should to satisfy the next necessary condition $2v_i^P(r) \leq \overline{x_i}^{*P}(r) - \underline{x}_i^{*P}(r)$, $i = 1, 2$, for each $r \in [0,1]$. In that sense, the obtained V^P is feasible, similarly, V^N is feasible, therefore, the unique solution of BFLS is $X = (x_1, x_2)^T$, given by:

$$\begin{aligned} x_1 &= \prec -2+r, 3-4r, -5-2s, 7+10s \succ, \\ x_2 &= \prec r, 2-r, 3-2s, 6+s \succ . \end{aligned}$$

References

[1] M. Akram, G. Muhammad, T. Allahviranloo, "Bipolar fuzzy linear system of equations", *Computational and Applied Mathematics* 38, article number 69, 2019.

[2] T. Allahviranloo, M. Ghanbari, "On the algebraic solution of fuzzy linear systems based on interval theory", *Applied Mathematical Modelling* 36, pp. 5360–5379, 2012.

[3] Đ. Dragić, B. Mihailović, Lj. Nedović, "The general algebraic solution of dual fuzzy linear systems", *Book of Abstracts of FSTA*, Liptovsky Jan, Slovak Republic, pp. 58–59, 2024.

[4] Đ. Dragić, B. Mihailović, Lj. Nedović, "The general algebraic solution of dual fuzzy linear systems and Stein fuzzy matrix equations", *Fuzzy Sets and Systems*, https://doi.org/10.1016/j.fss.2024.108997

[5] M. Friedman, M. Ming, A. Kandel, "Fuzzy linear systems", *Fuzzy Sets and Systems* 96, pp. 201–209, 1998.

[6] B. Mihailović, V. Miler Jerković, B. Malešević, "Solving fuzzy linear systems using a block representation of generalized inverses: The Moore-Penrose inverse", *Fuzzy Sets and Systems* 353, pp. 44–65, 2018.

[7] B. Mihailović, V. Miler Jerković, B. Malešević, "Solving fuzzy linear systems using a block representation of generalized inverses: The group inverse", *Fuzzy Sets and Systems* 353, pp. 66-85, 2018.

[8] V. Miler Jerković, B. Mihailović, B. Malešević, "The general algebraic solution of fuzzy linear systems based on a block representation of 1-inverses", *Iranian Journal of Fuzzy Systems* 20 (3), pp. 115-126, 2023.