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ON A SIMPLE SCALING CONDITION FOR $H-$MATRICES AND APPLICATIONS [1]

Maja Nedović [2] and Srđan Milićević [3]

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Abstract. In this paper we consider a simple scaling condition for a special subclass of $H-$matrices and we present applications of this condition in eigenvalue localization for Schur complements of matrices in this subclass. Spectra localizations for the Schur complement matrix are based on the construction of the diagonal scaling matrix for the given $H-$matrix and use only the entries of the original matrix.

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1. Introduction

The theory of $H-$matrices, together with related knowledge on classes of $M-$matrices and $P-$matrices, represents a research area of interest for mathematicians as well as for researchers in the field of economy, ecology, engineering. It is well-known that investigation of some special subclasses of $H-$matrices brought to light different possibilities for localizing spectra of square complex matrices in general. Spectra localizations provide important information on stability of dynamical systems. Also, $H-$matrices and related classes play an important role in research on existence and uniqueness of solutions in linear complementarity problems, in construction of iterative procedures for solving these problems and in error analysis. Schur complement appears in block-Gaussian elimination and proved to be useful in reducing the dimension of the problem in solving linear systems of equations. When considering Schur complements of $H-$matrices, in recent years different authors provided results on closure properties of some special subclasses of $H-$matrices under Schur complement. Also, dominance degree of Schur complement was discussed and compared to the dominance degree of the original matrix.

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[2]Department of Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, e-mail: maja.nedovic@uns.ac.rs

[3]Department of Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, e-mail: srdjan88@uns.ac.rs

In this paper we deal with a special subclass of $H-$matrices defined by a simple condition based on scaling. We provide information on eigenvalues of Schur complements for matrices in this class.

2. Special $H-$matrices

In this section we recall results on SDD matrices and $H-$matrices and deal with a simple condition defining $SDD-$scal matrices.

A matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is a strictly diagonally dominant (SDD) matrix if

$$|a_{ii}| > r_i(A), \text{ for all } i \in N = \{1, 2, ..., n\},$$

with $r_i(A)$ being a deleted row sum defined as follows

$$r_i(A) = \sum_{j \in N \setminus \{i\}} |a_{ij}|.$$

This well-known class of non-singular matrices is the starting point for research on wider classes of special $H-$matrices. Its main advantage is that it is defined by a very simple condition, easily checkable, with low calculation cost.

Although there are many different ways to introduce non-singular $H-$matrices, see [1], the scaling characterization given by Fiedler and Pták, see [8], is the most revealing for the subject of this paper. According to [8], a given matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an H-matrix if and only if there exists a diagonal non-singular matrix D such that AD is an SDD matrix. Moreover, we can always assume that D has only positive diagonal entries.

In literature, there are several subclasses of $H-$matrices that are introduced or further researched through a construction of special diagonal scaling matrices, see [3, 7, 17, 18]. We apply the type of scaling from [3] in the next section in order to obtain spectra information for Schur complements.

Let us consider a different type of row sums for the given complex square matrix with nonzero diagonal entries, as follows. If $a_{ii} \neq 0,\ i \in N$, define

$$R_i(A) = \sum_{k \in N \setminus \{i\}} \frac{r_k(A)}{|a_{kk}|} |a_{ik}|.$$

Let $A \in \mathbb{C}^{n,n},\ n \geq 2$, be a matrix with nonzero diagonal entries and let

$$r_i(A) > R_i(A),$$

for all $i \in N$. We call matrices satisfying this condition $SDD-$scal matrices. It is easy to see that any $SDD-$scal matrix is an $H-$matrix. Namely, define a non-singular diagonal matrix

$$D = diag(d_k),\ k = 1, ..., n,$$

with

$$d_k = \frac{r_k(A)}{|a_{kk}|},\ k = 1, ..., n.$$

Let us consider the matrix AD. It holds that

$$(AD)_{ii} = r_i(A),\ i = 1, ..., n,$$

$$r_i(AD) = \sum_{k \in N\setminus\{i\}} \frac{r_k(A)}{|a_{kk}|}|a_{ik}| = R_i(A),\ i = 1, ..., n.$$

Therefore, as A is an $SDD-$scal matrix, AD is an SDD matrix, implying that A can be scaled to SDD matrix by a non-singular diagonal matrix D. This proves that A is an $H-$matrix.

Although this condition represents a simple modification of SDD condition, it is easy to see that the class of $SDD-$scal matrices is not a subclass of SDD, nor SDD class is a subclass of $SDD-$scal. For example, if we consider any SDD matrix with at least one deleted row sum equal to zero, it is easy to notice that such matrix is not $SDD-$scal. On the other hand, there are matrices that belong to $SDD-$scal class, but do not belong to SDD, such as the matrix B, while the intersection of these classes is not empty, as there exist matrices that belong both to SDD and $SDD-$scal, such as the matrix C,

$$B = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 1 & 0 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 2 \end{bmatrix}.$$

Therefore, classes SDD and $SDD-$scal stand in a general relation. If we consider the recently introduced class of SDD_1 matrices, see [14, 16], that contains SDD class, it is easy to see that B does not belong to SDD_1, meaning that $SDD-$scal and SDD_1 also stand in a general relation.

Recall that the famous Geršgorin theorem, that provides an eigenvalue localization set for an arbitrary complex square matrix, is equivalent to the statement that every SDD matrix is non-singular. Geršgorin localization set is the union of n Geršgorin disks, one for each row of the given matrix, defined as $\Gamma_i(A) = \{z \in \mathbb{C} \,|\, |z - a_{ii}| \le r_i(A)\},\ i = 1, 2, ..., n$. For more details on eigenvalue localization sets see [18]. In the next section, we apply Geršgorin localization set and the construction of diagonal scaling matrices for $SDD-$scal matrices in order to define areas that include (or exclude) spectra of Schur complements.

3. Eigenvalues of Schur complements of $SDD-$scal matrices

Spectra localizations and closure properties for Schur complements of some special matrices were researched in [4, 5, 6, 9, 10, 13, 15, 17].

The Schur complement of A with respect to a proper subset of N, α, is denoted by A/α and defined as

$$A(\overline{\alpha}) - A(\overline{\alpha}, \alpha)(A(\alpha))^{-1}A(\alpha, \overline{\alpha})$$

where $A(\alpha, \beta)$ stands for the submatrix of $A \in \mathbb{C}^{n,n}$ consisting of the rows indexed by α and the columns indexed by β, while $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$. We assume $A(\alpha)$ to be a nonsingular matrix. For details on applications of Schur complement matrices see [2, 19]. In the remainder of this section, we are interested in providing information on eigenvalues of Schur complements without calculating Schur complements. It is, off course, possible to apply well-known results on localizing spectra once the Schur complement matrix is calculated using the entries of Schur complement matrix. However, in what follows, we provide preliminary information on the eigenvalues of Schur complements using only the entries of the original matrix. These preliminary bounds can be useful when solving large scale systems of linear equations via Schur-based iteration, as the concentration of eigenvalues could predict faster convergence.

In [12], the following result is obtained. If a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ is an SDD matrix with real diagonal entries, and α a proper subset of the index set, then, A/α and $A(\overline{\alpha})$ have the same number of eigenvalues whose real parts are greater (less) than $w(A)$ (resp. $-w(A)$), with

$$w(A) = \min_{j\in\overline{\alpha}} \left[|a_{jj}| - r_j(A) + \min_{i\in\alpha} \frac{|a_{ii}| - r_i(A)}{|a_{ii}|} \sum_{k\in\alpha} |a_{jk}| \right].$$

In [11], one can find another result on the dominant degree and the spectra localization for the Schur complement. If $A \in \mathbb{C}^{n,n}$ and $\alpha = \{i_1, i_2, \ldots, i_k\} \subseteq N_2(A) = \{i \in N : \ |a_{ii}| > r_i(A)\}$, $\overline{\alpha} = \{j_1, j_2, \ldots, j_l\}$, then, for every eigenvalue λ of A/α, there exists $1 \le t \le l$ such that

$$|\lambda - a_{j_t j_t}| \le r_{j_t}(A).$$

This means that the localization area for eigenvalues of A/α can be obtained by taking the union of those Geršgorin disks, formed for the matrix A, whose indices are in $\overline{\alpha}$.

Now we present main results on eigenvalue localization for Schur complements of $SDD-$scal matrices.

Theorem 3.1. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be an $SDD-$scal matrix with real diagonal entries and let α be a proper subset of the index set N. Then, A/α and $A(\overline{\alpha})$ have the same number of eigenvalues whose real parts are greater (less) than $w(D^{-1}AD)$ (resp. $-w(D^{-1}AD)$), where $D = diag(d_1, d_2, ..., d_n)$, $d_k = \dfrac{r_k(A)}{|a_{kk}|}$, $k = 1, ..., n$.*

Proof. Notice that $D^{-1}AD$ is an SDD matrix with real diagonal entries. For α being a proper subset of the index set N, it holds that

$$(D^{-1}AD)/\alpha = D^{-1}(\overline{\alpha})(A/\alpha)D(\overline{\alpha}),$$

which is similar to A/α. As $(D^{-1}AD)(\overline{\alpha})$ and $A(\overline{\alpha})$ are similar for any choice of α, their spectra coincide. Applying result from [12] to SDD matrix $D^{-1}AD$ we obtain that A/α and $A(\overline{\alpha})$ have the same number of eigenvalues whose real parts are greater (less) than $w(D^{-1}AD)$ (resp. $-w(D^{-1}AD)$). □

When real parts of all the eigenvalues of the matrix $A(\overline{\alpha})$ lie out of the interval $(-w(D^{-1}AD), w(D^{-1}AD))$, this vertical band represents the exclusion area for the spectrum of A/α.

The next result gives a preliminary localization for the spectrum of the Schur complement matrix through a modification of the Geršgorin set of the original matrix.

Theorem 3.2. *Let $A = [a_{ij}] \in \mathbb{C}^{n,n}$ be an $SDD-$scal matrix, let α be a subset of the index set N, and let $D = diag(d_1, d_2, ..., d_n)$, $d_k = \dfrac{r_k(A)}{|a_{kk}|}$, $k = 1, ..., n$. Then,*

$$\sigma(A/\alpha) = \sigma((D^{-1}AD)/\alpha) \subseteq \bigcup_{j\in\overline{\alpha}} \Gamma_j(D^{-1}AD).$$

Proof. It is easy to see that

$$(D^{-1}AD)/\alpha = D^{-1}(\overline{\alpha})(A/\alpha)D(\overline{\alpha}).$$

Therefore the matrices $(D^{-1}AD)/\alpha$ and A/α are similar. This implies that

$$\sigma(A/\alpha) = \sigma((D^{-1}AD)/\alpha).$$

As $D^{-1}AD$ is an SDD matrix, applying the result from [11] to the matrix $D^{-1}AD$, we prove our statement. □

The benefit of this result is that it allows us to construct a localization area for the spectrum of Schur complement matrix A/α by scaling those Geršgorin disks of the original matrix A that correspond to $\overline{\alpha}$. Notice that results of this type are obtained for $PH-$matrices, see [13], for Partition-Nekrasov matrices, see [17] and for SDD_1 matrices, see [14], only with different constructions of corresponding scaling matrices.

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References

[1] A. Berman and R. J. Plemmons, "Nonnegative Matrices in the Mathematical Sciences", *Philadelphia: Classics in Applied Mathematics* 9, SIAM, 1994.

[2] D. Carlson and T. Markham, "Schur complements of diagonally dominant matrices", *Czech. Math. J.* 29 (104), pp. 246-251, 1979.

[3] Lj. Cvetković and V. Kostić, "New criteria for identifying $H-$matrices", *J. Comput. Math.* 180, pp. 265-278, 2005.

[4] Lj. Cvetković, V. Kostić, M. Kovačević, and T. Szulc, "Further results on $H-$matrices and their Schur complements", *Appl. Math. Comput.* 198, pp. 506-510, 2008.

[5] Lj. Cvetković and M. Nedović, "Special $H-$matrices and their Schur and diagonal-Schur complements", *Appl. Math. Comput.* 208, pp. 225-230, 2009.

[6] Lj. Cvetković and M. Nedović, "Eigenvalue localization refinements for the Schur complement", *Appl. Math. Comput.* 218 (17), pp. 8341-8346, 2012.

[7] T.-B., Gan, T.-Z., Huang, "Simple criteria for nonsingular $H-$matrices", *Linear Algebra Appl.* 374, pp. 317-326, 2003.

[8] M. Fiedler and V. Pták, "On matrices with nonpositive off-diagonal elements and positive principal minors", *Czechoslovak Math. J.* 12, pp. 382-400, 1962.

[9] J. Liu, Y. Huang, and F. Zhang, "The Schur complements of generalized doubly diagonally dominant matrices", *Linear Algebra Appl.* 378, pp. 231-244, 2004.

[10] J. Liu and Y. Huang, "Some properties on Schur complements of $H-$matrices and diagonally dominant matrices", *Linear Algebra Appl.* 389, pp. 365-380, 2004.

[11] J. Liu, Y. Huang, and F. Zhang, "The dominant degree and the disc theorem for the Schur complement of matrix", *Appl. Math. Comput.* 215, pp. 4055-4066, 2010.

[12] J. Z. Liu and F. Z. Zhang, "Disc separation of the Schur complements of diagonally dominant matrices and determinantal bounds", *SIAM J. Matrix Anal. Appl.* 27 (3), pp. 665-674, 2005.

[13] M. Nedović and Lj. Cvetković, "The Schur complement of $PH-$matrices", *Applied Mathematics and Computation*, vol. 362, Article Number 124541, 2019.

[14] M. Nedović and I. Prodanović, "Spectra localization for Schur complements of SDD_1 matrices", *META 2023, The 8th Conference on Mathematics in Engineering: Theory and Applications, Novi Sad, Serbia, May 26-28*, pp. 31-36, 2023.

[15] J. M. Peña, "Eigenvalue localization and pivoting strategies for Gaussian elimination", *Appl. Math. Comput.* 219 (14), pp. 7725-7729, 2013.

[16] J. M. Peña, "Diagonal dominance, Schur complements and some classes of H-matrices and P-matrices", *Adv. Comput. Math* 35, pp. 357-373, 2011.

[17] T. Szulc, Lj. Cvetković and M. Nedović, "Scaling technique for Partition-Nekrasov matrices", *Appl.Math.Comput.* vol. 271C, pp. 201-208, 2015.

[18] R. S. Varga, "Geršgorin and his Circles", *Series in Computational Mathematics*, 36, Springer, 2004.

[19] F. Zhang (Ed.), *The Schur Complement and its Applications*, Springer, New York, 2005.