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STABILITY ANALYSIS OF A NANO BEAM WITH NONSYMMETRIC BOUNDARY CONDITIONS

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Abstract. In this paper we investigate stability of an axially loaded nano beam that is clamped at one end and elastically restrained against rotation on the other. We analyze elastically buckling nano beam based on Eringen's nonlocal elasticity theory. The Euler method of adjacent equilibrium configuration is used to derive the nonlinear governing equations. The critical axial force and postbuckling shape are obtained for the beam with the unit cross-sectional area. New numerical results are obtained. The numerical analysis includes the influence of the characteristic parameter of the small scale length on the critical load and the postbuckling shape.

AMS Mathematics Subject Classification (2020): 74G60

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1. Introduction

Recently, there has been significant research attention on structures at very small length scales. Nano rods, nano beams, and nano plates have particularly attracted interest due to their technical applications as nanoactuators, nanosensors, and electrochemical sensors. The classical local continuum theory is not entirely applicable to microstructures, especially nanostructures. In particular in cases where the influence of small-scale effects becomes more significant and cannot be ignored. The nonlocal continuum theory was introduced by Eringen [1] to incorporate an internal length scale. According to this theory, the stress tensor at a reference point in an elastic continuum depends on the strain field at all points within the domain. Peddieson et al. [2] developed a nonlocal Euler-Bernoulli beam model in their research. The nonlocal continuum theory has been utilized in numerous studies to model nano beams ([3], [4], [5], [6], [7]).

Researchers have developed various nonlocal beam models to investigate the behavior of nano beams. Wang and Lee [8] formulated the nonlocal theory for both Timoshenko and Euler-Bernoulli beams. Their results provided the

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first insights into the influence of nonlocal elasticity on the bending of nano beams, considering concentrated loads as Dirac delta functions. Additionally, Tuna and Kirca [9] derived the exact solution of the integral form of Eringen's nonlocal model for bending analysis of Euler-Bernoulli and Timoshenko beams. In [10], the static deflection and critical buckling load of nano beams were studied for different boundary conditions, taking into account varying nonlocal and material-distribution parameters.

The influence of nonlocal parameters on critical load levels, post-critical shapes, and the effect of various boundary conditions on the stability of nano beams under conservative loading was analyzed in studies conducted by [11].

The aim of this research is to analyze the stability and determine the post-critical shape of a nano beam with nonsymmetrical boundary conditions which have not been used before. The beam is clamped on one end and elastically restrained against rotation on the other end.

2. Mathematical formulation

Consider a straight nano beam of length L loaded by an axial force F with the action line coinciding with the x axis of a rectangular coordinate system $x - B - y$ (see Fig. 1). The beam is clamped at one end and elastically restrained against rotation on the other end (linear rotational spring), with the end C having the possibility of sliding along the x -axis. At the end C the beam is loaded by a compressive force F.

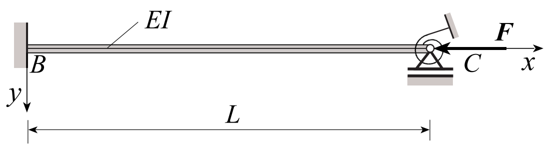

Figure 1: Coordinate system and load configuration

Equilibrium equations for the beam are (see [12])

$$\frac{dH}{dS} = 0, \quad \frac{dV}{dS} = 0, \quad \frac{dM}{dS} = -V\cos\theta + H\sin\theta, \tag{2.1}$$

where H and V are components of the contact force (i.e. the resultant force in an arbitrary cross-section) along x and y axes, respectively, M is the bending moment, θ is the angle between the tangent to the column axis and the x- axis of a rectangular Cartesian coordinate system $x - B - y$, S is the arc-length of the column axis measured from the origin of the coordinate system B. We adjoin to (2.1) the geometrical equations

$$\frac{d\bar{x}}{dS} = \cos\theta, \quad \frac{d\bar{y}}{dS} = \sin\theta, \tag{2.2}$$

and the constitutive equation for nonlocal beam theory

$$M - l^2\frac{d^2M}{dS^2} = EI\frac{d\theta}{dS}. \tag{2.3}$$

where l is an additional length scale specific to nonlocal constitutive law, that is, to a specific material. The value of l can be identified from the Born–Kármán model of lattice dynamics (see [1]). It can be observed from the experimental interpolations that $l \leq 2$ nm (nanometers). In (2.3) E is the modulus of elasticity and I is the moment of inertia of the cross section. In Equations (2.2) we used $\bar{x}$ and $\bar{y}$ to denote coordinates of an arbitrary point on the bear axis in the coordinate system $x - B - y$. If l=0 Equation (2.3) corresponds to the classical Bernoulli-Euler rod theory.

The boundary conditions for the column shown in Fig. 1 are

(2.4)
$$\bar{x}(0) = 0, \quad \bar{y}(0) = \bar{y}(L) = 0, \quad \theta(0) = 0, \quad M(L) = -c\theta(L), \quad H(L) = -F.$$

where c is a spring constant of the support and $c \neq \infty$. Solving $(2.1)_1$ and by using $(2.4)_5$, we obtain $H = -F$. By introducing the dimensionless quantities

(2.5)
$$t = \frac{S}{L},\ \zeta = \frac{\bar{x}}{L},\ \eta = \frac{\bar{y}}{L},\ k = \frac{l}{L},\ \lambda = \frac{FL^2}{EI},\ v = \frac{VL^2}{EI},\ m = \frac{ML}{EI},\ b = \frac{cL}{EI}$$

we obtain from (2.1)-(2.3)

$$\dot{v} = 0, \quad \dot{m} = -v\cos\theta - \lambda\sin\theta, \quad \dot{\zeta} = \cos\theta, \quad \dot{\eta} = \sin\theta, \quad \dot{\theta} = m - k^2\ddot{m}, \tag{2.6}$$

subject to

$$\eta(0) = \eta(1) = 0, \quad \theta(0) = 0, \quad m(1) = -b\theta(1), \tag{2.7}$$

where $\dot{(\cdot)} = \frac{d}{dt}(\cdot)$.

The trivial solution for the systems (2.6), (2.7) in which the axis of the rod remains straight for any value of and the dimensionless load parameter is

$$\theta_0 = v_0 = \eta_0 = 0, \quad \zeta_0 = t. \tag{2.8}$$

Euler method is used to examine stability of the trivial configuration defined by Equations (2.8) (see [12]). In order to obtain nontrivial solution to (2.6), (2.7) and determine $\lambda \in \mathbb{R}$ for it, it is assumed that

$$\theta = \theta_0 + \Delta\theta, \quad v = v_0 + \Delta v, \quad \eta = \eta_0 + \Delta\eta, \quad \zeta = \zeta_0 + \Delta\zeta, \tag{2.9}$$

where $\Delta\theta, ..., \Delta\zeta$ are perturbations. After substituting this in Equations (2.6) and (2.7) and by neglecting the higher order terms in perturbations, Equations (2.6) become (omitting Δ in front of variables) linearized equations describing relative equilibrium of the beam

$$\dot{v} = 0, \quad \dot{m} = -v - \lambda\theta, \quad \dot{\zeta} = 1, \quad \dot{\eta} = \theta, \quad \dot{\theta} = m - k^2\ddot{m}, \tag{2.10}$$

subject to (2.7).

3. Critical values of load parameter

In this section we will determine the critical loads (λ) of the beam for which the beam loses its stability, e.i. when system (2.6), (2.7) has nontrivial solution. A necessary condition for this to occur is that the linearized systems (2.10), (2.7) have a nontrivial solution.

The system (2.10) can be reduced, so it can be written as

$$\ddddot{\eta} + \frac{\lambda}{1-k^2\lambda}\ddot{\eta} = 0 \tag{3.1}$$

Then the solution of Equations (3.1) is

$$\eta = C_1 \cos(\beta\zeta) + C_2 \sin(\beta\zeta) + C_3\zeta + C_4, \tag{3.2}$$

where C_j, $j = 1, 2, 3, 4$are arbitrary constants and

$$\beta = \sqrt{\frac{\lambda}{1-k^2\lambda}}. \tag{3.3}$$

The boundary conditions subjected to Equations (3.2) are

$$\eta(0) = \eta(1) = 0, \quad \dot{\eta}(0) = 0, \quad \ddot{\eta}(1) = -b\dot{\eta}(1). \tag{3.4}$$

By using boundary conditions (3.4) the following condition for the existence of non-trivial solutions is derived

$$2b - \left(\beta^2 + 2b\right)\cos\beta + (\beta - b\beta)\sin\beta = 0. \tag{3.5}$$

The necessary condition for the existence of solutions of the Equations (3.5) is $k^2\lambda \leq 1$.

4. Numerical results

The critical (smallest positive root of (3.5)) value of the axial force for several values of k and b is determined from Equations (3.5) and shown in Table 1. Case when parameter b=∞ represents the nano beam that is clamped on the both ends.

Table 1. Critical values of λ for different values of k and parameter b

b	0	0.5	50	500
$k = 0$	20.19072856	21.65942569	29.57476388	39.32098463
$k = 0.05$	19.22053695	20.54684361	27.5386418	35.80159929
$k = 0.1$	16.79890687	17.80332726	22.82447831	28.22330372
$k = 0.2$	11.1697295	11.60506445	13.54782035	19.07257867

The value of critical axial force increases for the increasing value of spring constant b and constant value of k. When the nonlocal parameter (k) increases

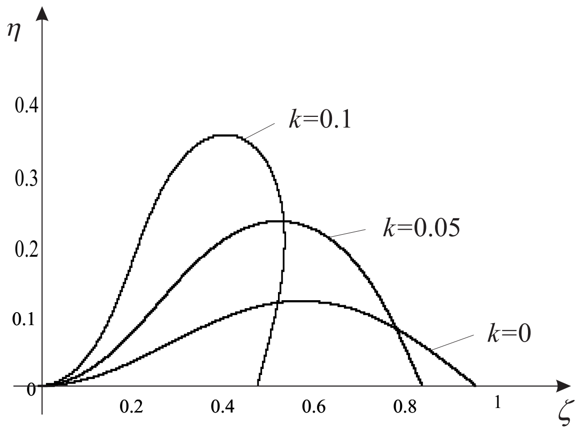

Figure 2: Postbuckling modes $\lambda = 22$, $b = 0.5$

than value of critical axial force also increases for constant value of b. For the case when $b = \infty$ and $k = 0$ (Bernoulli–Euler clamped beam) we have $\lambda = 4\pi^2 = 39.4784176$ (see [12]).

The postbuckling shapes of first modes for chosen $\lambda = 22$, $b = 0.5$ and several values of k are shown in Fig. 2. Equations (2.6) and (2.7) are used for determining postbuckling shapes.

5. Conclusions

In this paper we analyzed the stability for an elastic nano beam. The beam is clamped on one end and elastically restrained against rotation on the other end. The characteristic equation (3.5) that determins critical loads for the nano beam with constant cross-section is derived. By using the characteristic equation we determined the lowest value of λ for several values of parameters b and k (nonlocal parameter). Numerical analysis demonstrates that the value of spring constant and nonlocal parameters have an impact on critical value of axial force. The postbuckling shapes of first modes for chosen values of λ and b and several values of parametar k is determined.

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