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PROBABILISTIC REASONING ABOUT TYPED PROGRAMS: TOWARDS COMPACTNESS

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Abstract. Reasoning with uncertainty has gained an important role in various fields of computer science, artificial intelligence and cognitive science, while it is underdeveloped in typed calculi. For this reason, we have investigated different approaches used to introduce probability into typed calculi. In this paper, we develop PCL logic, which is a formal model for probabilistic reasoning about typed programs. The semantics of PCL is based on the possible world approach. Allowing the range of probability functions to be infinite results in non-compactness of the logic. As a consequence, a finite axiomatization of the logic cannot be sound and strongly complete. We propose the simplest method for resolving the non-compactness phenomenon of the logic PCL.

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1. Introduction

The motivation for developing a new formal model for reasoning about typed terms is the fact that reasoning with uncertainty has gained an important role in various fields of computer science, artificial intelligence and cognitive science, while it is underdeveloped in typed calculi.

Introducing non-determinism and probabilities into the typed calculi has been topic of several papers, e.g. [1, 2, 3, 4, 8]. Nevertheless, the goal of these papers was to formalize computation in the presence of uncertainty and not to provide a framework that enables probabilistic reasoning about typed terms. Our goal is to introduce the logic in which we can express the following sentence:

The probability that a term M inhabits a type σ is at least s.

We follow the method used for the logic LPP_2 [9], which is a probabilistic extension of the classical propositional logic. The language of the logic LPP_2

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is obtained by extending the language of the classical propositional logic with probability operators $P_{\geq s}$ whose meaning is "the probability is at least s".

The idea of formalization of probabilistic reasoning about typed terms has been introduced in [5, 6], where we have tried to developed the formal model upon the well-known models of λ-calculus and combinatory logic. However, these models are not suitable for propositional reasoning about typed terms. For this reason, we have first developed the logic LCL, a classical propositional extension of the simply typed combinatory logic, in [7]. Then, we have introduced the probability extension of the logic LCL in [10], and we present this extension in the next section.

2. The logic PCL

In this section, we introduce the syntax and semantics of the logic PCL. The logic PCL is a probabilistic extension of the logic LCL introduced in [7]. Hence, the language of PCL is layered into two sets of formulas: basic formulas (LCL-formulas) and probabilistic formulas.

The LCL-formulas are generated by the following grammar

$$\alpha := M : \sigma \mid \neg\alpha \mid \alpha \Rightarrow \alpha$$

where M is a term and σ a type such that there exists a basis Γ in which M inhabits σ.

Probabilistic formulas are obtained by applying the probability operator to LCL-formulas. For $s \in [0, 1] \cap \mathbb{Q}$ and an LCL-formula α, the formula $P_{\geq s}\alpha$ is called *a basic probabilistic formula.* The set of all probabilistic formulas is generated by the following grammar

$$\varphi := P_{\geq s}\alpha \mid \neg\varphi \mid \varphi \wedge \varphi$$

The language of PCL is the union of basic and probabilistic formulas. The set of all PCL-formulas is denoted by For_{PCL} and is ranged over by $A, B, C, \ldots$.

Although we have used only negation and implication in the definition of LCL-formulas, and negation and conjunction in the definition of probabilistic formulas, other classical propositional connectives are defined as usual. Please notice that mixing of basic formulas and probabilistic formulas, and nested probability operators are not allowed

Following the approach used for the logic LPP_2 and other probability logics [9], the semantics of the logic PCL is based on the possible world approach, where the set of possible worlds is equipped with a finitely additive probability measure.

Definition 2.1. A PCL-model is a structure

$$\mathcal{M} = (W, \{D_w\}, \{A^\sigma_w\}, \{\cdot_w\}, \{\mathbf{s}_w\}, \{\mathbf{k}_w\}, \{\mathbf{i}_w\}, H, \mu, \rho)$$

where:

- W is a non-empty set of objects, called *possible worlds*,

- $\{D_w\} = \{D_w\}_{w \in W}$ is a family of sets indexed by worlds, where the set D_w is referred to as *the domain* of the world w,
- $\{A_w^\sigma\} = \{A_w^\sigma\}_{w \in W, \sigma \in \mathsf{Types}_\to}$ is a family of sets indexed by types σ and worlds w such that $A_w^\sigma \subseteq D_w$ for all $w \in W$ and $\sigma \in \mathsf{Types}_\to$.
- $\{\cdot_w\} = \{\cdot_w\}_{w \in W}$ is a family of binary operations indexed by worlds such that the following hold:
  - $\cdot_w$ is a binary operation on D_w, i.e. $\cdot_w : D_w \times D_w \to D_w$,
  - $\cdot_w$ is extensional, that is for every $w \in W$ and every $d_1, d_2 \in D_w$, if $(\forall e \in D_w)(d_1 \cdot_w e = d_2 \cdot_w e)$, then $d_1 = d_2$,
  - for every $\sigma, \tau \in \mathsf{Types}_\to$, it holds that the codomain of the restriction of function $\cdot_w$ to the set $A_w^{\sigma\to\tau} \times A_w^\sigma$ is A_w^τ,
- $\{\mathbf{s}_w\} = \{\mathbf{s}_w\}_{w \in W}$ is a family of elements indexed by worlds such that for every $w \in W$ the following hold:
  - $\mathbf{s}_w \in D_w$,
  - for every $\sigma, \tau, \rho \in \mathsf{Types}_\to$, $\mathbf{s}_w \in A_w^{(\sigma\to(\tau\to\rho))\to((\sigma\to\tau)\to(\sigma\to\rho))}$
  - and for every $d, e, f \in D_w$, $((\mathbf{s}_w \cdot_w d) \cdot_w e) \cdot_w f = (d \cdot_w f) \cdot (e \cdot_w f)$
- $\{\mathbf{k}_w\} = \{\mathbf{k}_w\}_{w \in W}$ is a family of elements indexed by worlds such that for every $w \in W$ the following hold:
  - $\mathbf{k}_w \in D_w$,
  - for every $\sigma, \tau \in \mathsf{Types}_\to$, $\mathbf{k}_w \in A_w^{\sigma\to(\tau\to\sigma)}$
  - and for every $d, e \in D_w$, $(\mathbf{k}_w \cdot_w d) \cdot_w e = d$
- $\{\mathbf{i}_w\} = \{\mathbf{i}_w\}_{w \in W}$ is a family of elements indexed by worlds such that for every $w \in W$ the following hold:
  - $\mathbf{i}_w \in D_w$,
  - for every $\sigma \in \mathsf{Types}_\to$, $\mathbf{i}_w \in A_w^{\sigma\to\sigma}$
  - and for every $d \in D_w$, $\mathbf{i}_w \cdot_w d = d$
- H is an algebra of subsets of W.
- μ is a finitely additive probability measure, $\mu : H \to [0, 1]$.
- $\rho : W \times V \to \bigcup_{w \in W} D_w$ provides for each world a valuation of term variables such that for every $w \in W$, $\rho(w, \cdot)$ is a map from the set of term variables to the domain D_w, i.e. $\rho(w, \cdot) : V \to D_w$.

The semantics of PCL is defined in such way that each world of a model represents one LCL-model. We formally state this in the next proposition.

Proposition 2.2. *Let*

$$\mathcal{M} = (W, \{D_w\}, \{A_w^\sigma\}, \{\cdot_w\}, \{\mathbf{s}_w\}, \{\mathbf{k}_w\}, \{\mathbf{i}_w\}, H, \mu, \rho)$$

be a PCL-model. For each $w \in W$, the structure

$$\mathcal{M}_w = \langle D_w, \{A_w^\sigma\}_\sigma, \cdot_w, \mathbf{s}_w, \mathbf{k}_w, \mathbf{i}_w\rangle$$

is an applicative structure for LCL and $\mathcal{M}_{\rho_w} = \langle \mathcal{M}_w, \rho(w, \cdot)\rangle$ is an LCL-model.

The satisfiability of a formula in a model is defined inductively. First, we introduce the notion of satisfiability of a basic formula α in a possible world w of a model $\mathcal{M}$.

Definition 2.3. Let $\mathcal{M} = (W, \{D_w\}, \{A_w^\sigma\}, \{\cdot_w\}, \{\mathbf{s}_w\}, \{\mathbf{k}_w\}, \{\mathbf{i}_w\}, H, \mu, \rho)$ be a PCL-moodel, w' a possible world in $\mathcal{M}$ and α a basic formula. The formula α is satisfied in a world w', denoted by $w' \models \alpha$, if and only if α is satisfied by the LCL-model $\mathcal{M}_{\rho_{w'}} = \langle \mathcal{M}_{w'}, \rho_{w'}\rangle$ where $\mathcal{M}_{w'} = \langle D_{w'}, \{A_{w'}^\sigma\}_\sigma, \cdot_{w'}, \mathbf{s}_{w'}, \mathbf{k}_{w'}, \mathbf{i}_{w'}\rangle$ and $\rho_{w'}(x) = \rho(w', x)$.

Now, the satisfiability of a probabilistic formula is defined only for the class of measurable PCL models as follows.

Definition 2.4. A PCL-model

$$\mathcal{M} = (W, \{D_w\}, \{A_w^\sigma\}, \{\cdot_w\}, \{\mathbf{s}_w\}, \{\mathbf{k}_w\}, \{\mathbf{i}_w\}, H, \mu, \rho)$$

is *measurable* if $[\alpha]_{\mathcal{M}} \in H$ for every formula $\alpha \in \mathsf{For}_{\mathsf{B}}$, where $[\alpha]_{\mathcal{M}} = \{w \in W \mid w \models \alpha\}$. The class of all measurable PCL-models is denoted by PCL_{Meas}.

Definition 2.5. The satisfiability relation $\models \subseteq PCL_{\mathsf{Meas}} \times \mathsf{For}_{PCL}$ is defined in the following way:

- $\mathcal{M} \models \alpha$ if and only if for every $w \in W$, $w \models \alpha$.
- $\mathcal{M} \models P_{\geq s}\alpha$ if and only if $\mu([\alpha]) \geq s$.
- $\mathcal{M} \models \neg\phi$ if and only if it is not the case that $\mathcal{M} \models \phi$.
- $\mathcal{M} \models \phi \wedge \psi$ if and only if $\mathcal{M} \models \phi$ and $\mathcal{M} \models \psi$.

3. Towards compactness

In this section, we deal with non-compactness of the logic PCL. We say that a logic L satisfies the compactness theorem if the following holds:

> The set X of formulas of the logic L is satisfiable if and only if every finite subset of X is satisfiable.

The logic PCL does not satisfy the compactness theorem. Let us consider the following set

$$X = \{\neg P_{=0}(x : \sigma)\} \cup \{P_{<\frac{1}{n}}(x : \sigma) \mid n \in \mathbb{N}\}.$$

The set X is not satisfiable, and every finite subset of X is satisfiable. For more details about proving satisfiability of finite subsets of X and non-satisfiability of the set X we refer the reader to [10].

As the consequence of non-compactness, we have that a finite axiomatization can not be sound and strongly complete. More precisely, the inconsistency of the set X can not be proved using finite proof. Let us assume that there is a finite axiomatization of PCL, which is sound and strongly complete and let X be an infinite set of formulas such that every subset of X is satisfiable and X itself is not. From the strong completeness of the axiomatization, it follows that the set X is inconsistent, since it is unsatisfiable. So, it holds that $X \vdash \perp$. Since the axiomatization is finite, the proof of $X \vdash \perp$ has to be a finite sequence of formulas. Thus, there exists a finite subset $X' \subseteq X$ such that $X' \vdash \perp$. Then X' is also inconsistent. Furthermore, we conclude X' is unsatisfiable by the soundness of the axiomatization. This contradicts the assumption that every finite subset of X is satisfiable. We see that if we take a finite strongly complete axiomatization, there will be unsatisfiable sets, that are consistent, which results in unsoundness of the axiomatization. In [10], we have introduced the infinite axiomatization for PCL, that has one infinitary rule (the rule with countably many premises). This rule corresponds to the Archimedean axiom for real numbers and it guarantees that the set X is inconsistent ($X \vdash \perp$).

Now, we focus on the method for resolving non-compactness. The simplest method for this is to allow only probability measures with fixed finite ranges in models. We follow the approach used in [9] to resolve the non-compactness phenomenon of the logic LPP_2, and we set the range of probability measure to be the set $Fr(n) = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$. The obtained logic is denoted by $PCL^{Fr(n)}$. So, there is countably many more logics (for each positive integer n, one logic) which are similar to the logic PCL.

With this change in the semantics, the set X introduced above is still unsatisfiable, but now it is possible to give a finitary strongly complete axiomatization such that X is inconsistent.

The next step is to give sound and strongly complete axiomatizations for the logics $PCL^{Fr(n)}$. Finally, after we obtain the soundness and completeness results for the logics $PCL^{Fr(n)}$ we plan to prove Compactness theorem for these logics.

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