The 9th Conference on Mathematics in Engineering: Theory and Applications
Novi Sad, May 31st–June 2nd 2024

A SHORT NOTE ON THE EXTENDED GEVREY REGULARITY

Filip Tomić [1]

https://doi.org/10.24867/META.2024.24 Review article

Abstract. In this paper we give an overview of the results concerning extended Gevrey classes. They are studied in detail by the author and his collaborators in the series of papers. Here we present an equivalent definition of such classes, recall some of their properties and briefly discuss some applications.

AMS Mathematics Subject Classification (2020): 46E10; 46F05

Key words and phrases: Gevery classes, weight sequences, associated function, ultradifferentable functions

1. Introduction

Classes of ultradifferentiable functions are determined by the sequences of positive numbers M_p, $p \in \mathbb{N}$. They are often called *weight sequences*, and they control the derivatives of the function. By imposing some technical conditions on M_p, one can prove different properties of the corresponding classes such as closedness under pointwise multiplication, closednes under composition or closedness under action of differential operators of infinite order. For instance, *moderate growth* condition

$$M_{p+q} \leq C M_p M_q, \quad C > 0, \quad p, q \in \mathbb{N}, \tag{1.1}$$

is important in contruction of differential oparators of infinite order (ultradifferentiable operators) which acts continuously on such spaces (see [5]). If $M_p = p^{tp}$, $t > 1$, we obtain well known Gevrey spaces (see [10]). For $t = 1$ we have spaces of locally analytic functions.

Extended Gevrey classes $\mathcal{E}_{\tau,\sigma}(U)$ are introduced in [7]. They contain functions whose derivativers are controlled by $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$. Such sequences do not satisfy the condition (1.1) for any choice of τ and σ, and therefore it is not possible to use arguments from the standard theory to analyze their properties. However, it turns out that the choice of two parameters τ and σ gives an advantage in a topological sense. More precisely, it is possible to define productive and inductive limit topologies on $\bigcap_{\tau>0} \mathcal{E}_{\tau,\sigma}(U)$ and $\bigcup_{\tau>0} \mathcal{E}_{\tau,\sigma}(U)$ prove that such classes have "nice" properties. For detail exposure on extended Gevrey classes we refer to [7, 8, 9, 11, 12].

[1] Department of Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, e-mail: filip.tomic@uns.ac.rs

The aim of this paper is to recall some of the basic properties of $\mathcal{E}_{\tau,\sigma}(U)$. We will also give an equivalent definition of such classes by using products of sequences that increases to infinity. At the end we will discuss some recent applications.

Let us start with preliminary notation.

1.1. Notation

We use the standard notation: $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{C}$, denote sets of positive integers, nonnegative integers, integers, real numbers, positive real numbers and complex numbers, respectively. The length of a multi-index $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$ is denoted by $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_d$ and $\alpha! := \alpha_1! \cdots \alpha_d!$. For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we denote: $|x| := \left(x_1^2 + \ldots + x_d^2\right)^{1/2}$, $x^\alpha := \prod_{j=1}^d x_j^{\alpha_j}$, and $\partial^\alpha = \partial_x^\alpha := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$, $j = 1, \dots, d$. We write $\lfloor x \rfloor := \max\{m \in \mathbb{N} : m \leq x\}$.

2. Extended Gevrey classes

We start with the properties of weight sequences $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $\tau > 0$, $\sigma > 1$, $p \in \mathbb{N}$, which are given in the following Lemma.

Lemma 2.1. *Let $\tau > 0$, $\sigma > 1$, $M_0^{\tau,\sigma} = 1$, and $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbb{N}$. Then the following properties hold:*

$(M.1)$ $(M_p^{\tau,\sigma})^2 \leq M_{p-1}^{\tau,\sigma} M_{p+1}^{\tau,\sigma}, \quad p \in \mathbb{N},$

$\widetilde{(M.2)}$ $M_{p+q}^{\tau,\sigma} \leq C^{p^\sigma + q^\sigma} M_p^{\tau 2^{\sigma-1},\sigma} M_q^{\tau 2^{\sigma-1},\sigma}, \; p, q \in \mathbb{N}_0,$ *for some constant $C \geq 1$,*

$\widetilde{(M.2)'}$ $M_{p+1}^{\tau,\sigma} \leq C^{p^\sigma} M_p^{\tau,\sigma}, \quad p \in \mathbb{N}_0,$ *for some constant $C \geq 1$,*

$(M.3)'$ $\displaystyle\sum_{p=1}^\infty \frac{M_{p-1}^{\tau,\sigma}}{M_p^{\tau,\sigma}} < \infty.$

Remark 2.1. Note that $M_p^{\tau,1} = p^{\tau p}$ are Gevrey sequences. They satisfy condition $(M.1)$ for any $\tau > 0$, and $(M.3)'$ only for $\tau > 1$. Moreover, conditions $\widetilde{(M.2)}$ and $\widetilde{(M.2)'}$ are classical Komatsu's conditions $(M.2)$ and $(M.2)'$, where $(M.2)$, given by (1.1), is stronger (see [5]).

When $\sigma > 1$ note that in $\widetilde{(M.2)}$ parameter τ increases to $\tau 2^{\sigma-1}$, $\tau > 0$. Therefore, $\widetilde{(M.2)}$ and $\widetilde{(M.2)'}$ are not comparable.

Now we can define extended Gevrey classes.

Definition 2.1. Let $\tau > 0$ and $\sigma > 1$. The extended Gevrey class $\mathcal{E}_{\tau,\sigma}(\mathbb{R}^d)$ is the set of all $\phi \in C^\infty(\mathbb{R}^d)$ such that for every compact set $K \subset\subset \mathbb{R}^d$ there exists constant $h > 0$ such that

$$|\partial^\alpha \phi(x)| \leq h^{|\alpha|^\sigma + 1} M_{|\alpha|}^{\tau,\sigma}, \quad x \in K, \quad \alpha \in \mathbb{N}_0^d. \tag{2.1}$$

Remark 2.2. Note that $\mathcal{E}_{t,1}(\mathbb{R}^d) := \mathcal{G}(\mathbb{R}^d)$, $t > 1$, and $\mathcal{E}_{1,1}(\mathbb{R}^d) := \mathcal{A}(\mathbb{R}^d)$ are Gevery class of order t and class of locally analytic functions on $\mathbb{R}^d$, respectively. Moreover, $\mathcal{E}_{\tau,1}(\mathbb{R}^d)$, $0 < \tau < 1$ are classes of quasi-analytic functions.

It is also possible to distinguish extended Gevrey classes of *Roumieu* and *Beurling* type. Here we will discuss only Rouimeu case. For details see [7].

Basic properties of classes $\mathcal{E}_{\tau,\sigma}(\mathbb{R}^d)$ are given in the following proposition. The proof is based on the properties of $M_p^{\tau,\sigma}$ given in Lemma 2.1.

Proposition 2.1. *([7, 8, 9])*

$i)$ *For* $\sigma_2 > \sigma_1 > 1$ *we have*

$$\mathcal{A}(\mathbb{R}^d) \subset \bigcup_{t>1} \mathcal{G}_t(\mathbb{R}^d) \subset \bigcup_{\tau>0} \mathcal{E}_{\tau,\sigma_1}(\mathbb{R}^d) \subset \bigcap_{\tau>0} \mathcal{E}_{\tau,\sigma_2}(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d),$$

with strict inclusions.

$ii)$ $\mathcal{E}_{\tau,\sigma}(\mathbb{R}^d)$ *is closed under the pointwise multiplication.*

$iii)$ $\mathcal{E}_{\tau,\sigma}(\mathbb{R}^d)$ *is closed under finite order derivation.*

$iv)$ $\mathcal{E}_{\tau,\sigma}(\mathbb{R}^d)$ *is closed under superposition. More precisely, if* F *is an entire function on* $\mathbb{R}$ *and* $f(x) \in \mathcal{E}_{\tau,\sigma}(\mathbb{R}^d)$ *then* $F(f(x)) \in \mathcal{E}_{\tau,\sigma}(\mathbb{R}^d)$.

$v)$ *Classes* $\bigcap_{\tau>0} \mathcal{E}_{\tau,\sigma}(\mathbb{R}^d)$ *and* $\bigcup_{\tau>0} \mathcal{E}_{\tau,\sigma}(\mathbb{R}^d)$ *are ultradifferentiable, i.e., for each class there exist a differential operator of infinite order which is continuous.*

In the classical theory (see [5]) expression h^{p^σ} appearing in (2.1) is usually written as h^p. This modification in our case is due to properties $\widetilde{(M.2)}$ and $\widetilde{(M.2)'}$ of $M_p^{\tau,\sigma}$. In the sequel we prove that h^{p^σ} in the definition of $\mathcal{E}_{\tau,\sigma}$ can be replaced by the product of sequences that increases to infinity.

We start with the following lemma.

Lemma 2.2. *Let* $\sigma = 2, 3, 4...$ *and* $a_p > 0$ *for* $p \in \mathbb{N}$.

i) There exists a constant $h > 0$ *such that*

$$\sup\left\{ \frac{a_p}{h^{p^\sigma}} : p \in \mathbb{N}_0 \right\} < \infty \tag{2.2}$$

if and only if

$$\sup\left\{ \frac{a_p}{R_{p,\sigma}} : p \in \mathbb{N}_0 \right\} < \infty \tag{2.3}$$

for arbitrary positive sequence (r_j) *that increase to infinity, where*

$$R_{0,\sigma} = 1, \quad R_{p,\sigma} := \prod_{j=1}^{p^\sigma} r_j \quad p \in \mathbb{N}. \tag{2.4}$$

ii) There exist positive sequence (r_j) *that increase to infinity such that*

$$\sup\{R_{p,\sigma} a_p : \quad p \in \mathbb{N}_0\} < \infty, \tag{2.5}$$

where R_p *is given by* (2.4)*, if and only if*

$$\sup\left\{ h^{p^\sigma} a_p : p \in \mathbb{N}_0 \right\} < \infty \tag{2.6}$$

for every $h > 0$.

Proof. $i)$ Let $a_p \leq Ch^{p^\sigma}$, $p \in \mathbb{N}_0$, for some $C, h > 0$, and let (r_j) be arbitrary sequence of positive numbers that increase to infinity. Choose j_0 such that $\frac{h}{r_j} \leq 1$, $j \geq j_0$. Then

$$a_p \leq Ch^{p^\sigma} = C \prod_{j=1}^{j_0} r_j \frac{h}{r_j} \prod_{j=j_0+1}^{p^\sigma} r_j \frac{h}{r_j} \leq CC_1^{j_0} \prod_{j=1}^{p^\sigma} r_j,$$

for large p and suitable $C_1 > 0$. This proves (2.3).

The opposite part we prove by contradiction. Assume that (2.3) holds for arbitrary (r_j), and that (2.2) is violated, i.e., $\sup\left\{\frac{a_p}{h^{p^\sigma}} : p \in \mathbb{N}_0\right\} = \infty$ for every $h > 0$. Arguing in a similar way as in [6] (see also [1]), we can choose a subsequence $(a_{p_m})_{m\in\mathbb{N}}$ of $(a_p)_{p\in\mathbb{N}}$ and strictly increasing sequence $(h_m)_{m\in\mathbb{N}}$ such that

$$p_{m+1} > p_m, \quad a_{p_m} > m\, h_m^{p_m^\sigma} \quad m \in \mathbb{N}_0.$$

Note that $p_{m+1}^\sigma > p_m^\sigma$ for all $\sigma = 2, 3, \ldots$.

Let us define a step sequence

$$r_j := h_1,\ 1 \leq j \leq p_1^\sigma, \quad r_j := \left(\frac{h_m^{p_m^\sigma}}{h_{m-1}^{p_{m-1}^\sigma}}\right)^{\frac{1}{p_m^\sigma - p_{m-1}^\sigma}},\ p_{m-1}^\sigma < j \leq p_m^\sigma;$$

where $m = 2, 3, \ldots$. Note that (r_j) increase to ∞, and

$$R_{p_m,\sigma} = \prod_{j=1}^{p_m^\sigma} r_j = \prod_{j=1}^{p_1^\sigma} h_1^{p_1^\sigma} \cdots \prod_{j=p_{m-1}^\sigma+1}^{p_m^\sigma} \left(\frac{h_m^{p_m^\sigma}}{h_{m-1}^{p_{m-1}^\sigma}}\right)^{\frac{1}{p_m^\sigma - p_{m-1}^\sigma}} = h_m^{p_m^\sigma}, \quad m \in \mathbb{N}_0.$$

Hence we obtain, $\frac{a_{p_m}}{R_{p_m,\sigma}} > m$ and therefore (2.3) is not satisfied.

$ii)$ The if part follows simlarly as in $i)$.

Let us assume now that condition (2.6) is satisfied for every $h > 0$. Put $C_h := \sup\left\{h^{p^\sigma} a_p : p \in N_0\right\}$ for $h \geq 1$. We define

$$H_j := \sup\left\{\frac{h^j}{C_h} : h \geq 1\right\}, \quad j \in \mathbb{N}_0.$$

Note that

$$H_{p^\sigma} a_p = \sup\left\{\frac{h^{p^\sigma} a_p}{C_h} : h \geq 1\right\} \leq 1.$$

From the arguments of [6, Lemma 3.4] it follows that H_j satisfies $(M.1)$ and H_j/h^j tends to infinity for all $h \geq 1$. Therefore we may choose $r_j = \frac{H_j}{H_{j-1}}$, $j \in \mathbb{N}$, to obtain

$$R_{p,\sigma}a_p = \Big(\prod_{j=1}^{p^\sigma} r_j\Big) a_p = H_{p^\sigma} a_p \leq 1,$$

and (2.5) follows.

□

As an easy cosequence of the previous Lemma we obtain the following Theorem.

Theorem 2.1. *Let $\tau > 0$ and $\sigma > 1$. A function $\phi \in C^\infty(\mathbb{R}^d)$ belongs to $\mathcal{E}_{\tau,\sigma}(\mathbb{R}^d)$ if and only if for any compact set K and any sequence (r_j) that increases to infinity there exists $C > 0$ such that*

$$|\partial^\alpha \phi(x)| \leq C R_{p,\sigma} M^{\tau,\sigma}_{|\alpha|}, \quad x \in K,\, \alpha \in \mathbb{N}_0,$$

where $R_{p,\sigma} = \prod_{j=1}^{\lfloor p^\sigma \rfloor} r_j$.

Proof. Without loss of generality we may assume that $h \geq 1$. From the simple inequality $\lfloor p^\sigma \rfloor \leq p^\sigma \leq 2\lfloor p^\sigma \rfloor$, $p \in \mathbb{N}$, it follows that $h^{\lfloor p^\sigma \rfloor} \leq h^{p^\sigma} \leq h^{2\lfloor p^\sigma \rfloor}$. Therefore (2.1) holds if and only if for every compact set $K \subset\subset \mathbb{R}^d$ there exists a constant $h_1 > 0$ such that

$$|\partial^\alpha \phi(x)| \leq h_1^{\lfloor |\alpha|^\sigma \rfloor + 1} M^{\tau,\sigma}_{|\alpha|}, \quad x \in K, \quad \alpha \in \mathbb{N}_0^d.$$

Now the statements follows from part $i)$ of the Lemma (2.2), putting $R_{p,\sigma} = \prod_{j=1}^{\lfloor p^\sigma \rfloor} r_j$ for $\sigma > 1$ instead of $R_{p,\sigma} = \prod_{j=1}^{p^\sigma} r_j$ for $\sigma = 2, 3, 4 \ldots$. □

Remark 2.3. Second part of the Lemma (2.2) is used to prove similar characterization in the Beurling case. We refer to [13] for details.

3. Applications

In [2] spaces $\mathcal{E}_{1,2}$ are used as an appropriate solution spaces for strictly hyperbolic equations with low regularity with respect to time variable. Recently it has been proved that extended Gevrey classes fits in the theory of weight matrix spaces. Namely, it is possible to define classes $\mathcal{E}_{\tau,\sigma}$ using familes of sequences, called *weight matrices*, of the form $\{\tau^{p^\sigma} p^{\tau p^\sigma}\}_{\tau>0}$ and to characterize them by the functions of the Braun-Meise-Taylor type. For the details we refer to see [4, 12]. Finally, in [3] properties of the sequences $M_p^{\tau,\sigma}$ are used to prove the surjectivity of Borel mappings.

Acknowledgement

The author acknowledge financial support of Department of Fundamental Sciences, Faculty of Technical Sciences, University of Novi Sad, in the frame of Project "Unapređenje nastavnog procesa na engleskom jeziku u opštim disciplinama", "Improving the teaching process in the English language in fundamental disciplines."

References

[1] R. Carmichael, A. Kaminski, S. Pilipović, *Notes on Boundary Values in Ultra-distribution Spaces*, Lecture Notes Series of Seul University, 49, 1999.

[2] M. Cicognani, D. Lorenz, ”Strictly hyperbolic equations with coefficients low-regular win time and smooth in space”, *J. Pseudo-Differ. Oper. Appl.*, 9, pp. 643–675, 2018.

[3] J. Jiménez-Garrido, A. Lastra, J. Sanz, ”Extension Operators for Some Ultra-holomorphic Classes Defined by Sequences of Rapid Growth”, *Constructive Approximation*, 2023. https://doi.org/10.1007/s00365-023-09663-z

[4] J. Jiménez-Garrido, D.N. Nenning, G. Schindl, ”On generalized definitions of ultradifferentiable classes”, *J. Math. Anal. Appl.* 526, 127260, 2023.

[5] H. Komatsu, ”Ultradistributions, I: Structure theorems and a characterization”, *J. Fac. Sci. Univ. Tokyo*, Sect. IA Math., 20 (1), pp. 25–105, 1973.

[6] H. Komatsu, ”Ultradistributions, III: Structure theorems and a characterization”, *J. Fac. Sci. Univ. Tokyo*, Sect. IA Math., 20 (1), 1973.

[7] S. Pilipović, N.Teofanov, F. Tomić, ”On a class of ultradifferentiable functions”, *Novi Sad Journal of Mathematics*, 45 (1), pp. 125–142, 2015.

[8] S. Pilipović, N.Teofanov, F. Tomić, ”Beyond Gevrey regularity: Superposition and propagation of singularities”, *Filomat*, 32 (8), pp. 2763–2782, 2018.

[9] S. Pilipović, N.Teofanov, F. Tomić, ”A Paley–Wiener theorem in extended Gevrey regularity”, *J. Pseudo-Differ. Oper. Appl.*, 11, pp. 593–612, 2020.

[10] L. Rodino, *Linear Partial Differential Operators in Gevrey Spaces*. World Scientific, 1993.

[11] N.Teofanov, F. Tomić, ”Inverse closedness and localization in extended Gevrey regularity”, *J. Pseudo-Differ. Oper. Appl.*, 8, pp. 411-421, 2017.

[12] N.Teofanov, F. Tomić, ”Extended Gevrey regularity via weight matrices”, *Axioms* 2022, 11, 576, 6pp, 2022.

[13] N.Teofanov, F. Tomić, M. Žigić, ”An introduction to Extended Gevrey regularity”, arXiv:2402.18341, 2024.