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ORTHOGONAL DECOMPOSITION OF STOCHASTIC TRANSPORT EQUATION

Stefan Tošić [1]

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Abstract. Uncertainty quantification seeks to provide a quantitative means to understand complex systems that are impacted by parametric uncertainty. The generalized polynomial chaos (gPC), see[3] and [2], as a tool of stochastic Galerkin method, is used for numerical solving of those complex systems which are described by stochastic ordinary differential equations (SODE) and stochastic partial differential equations (SPDE). We take transport equation as an example, and show that the Galerkin procedure results in a system of deterministic partial differential equations whose solving require an additional effort. Moreover, taking as a model various examples of SPDEs, numerical results with appropriate errors and deviations from exact solutions are presented. To show these results in a clearly visible manner, we used a number of figures, obtained using programming language Matlab.

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1. General approach

Let us consider general form of prominent transport stochastic partial differential equation, given by

$$\frac{du(x,t,Z)}{dt} = C(Z)\frac{du(x,t,Z)}{dx} + f(x,t,Z), \quad x \in (-1,1), \quad t > 0, \tag{1.1}$$

with initial condition,

$$u(x,0,Z) = u_0(x,Z). \tag{1.2}$$

and boundary conditions

$$\begin{aligned} u(1,t,Z) =& u_+(t,Z), \qquad C(Z) > 0, \\ u(-1,t,Z) =& u_-(t,Z), \qquad C(Z) < 0. \end{aligned} \tag{1.3}$$

It is important for further work to say that x is spatial variable, t is time, and $C(Z) \in L^2(\Omega)$ is an arbitrary function in terms of a random variable Z.

[1]Faculty of Technical Sciences, University of Novi Sad, e-mail: stefan.tosic@uns.ac.rs

By theory of generalized polynomial chaos(gPC) and Galerkin method(see [1] and [4]), approximation of solution of the governing equation has following form,

$$u(x,t,Z) \approx v_N(x,t,Z) = \sum_{k=0}^{P} v_k(x,t)\Phi_k(Z), \qquad P = \binom{N+d}{N}, \tag{1.4}$$

where N is order of gPC expansion, and $\{\Phi_k\}$ is d-variate gPC basis of orthogonal polynomials.
By substituting (1.4) into the governing equation, and taking projection of

$$\frac{dv_N(x,t,Z)}{dt} = C(Z)\frac{dv_N(x,t,Z)}{dx} + f(x,t,Z), \tag{1.5}$$

onto space spanned by $\{\Phi_k(Z)\}$, we get

$$\frac{dv_k(x,t)}{dt} = \sum_{i=0}^{P} a_{ik}\frac{dv_i(x,t)}{dx} + f_k(x,t), \quad k = 0,1,\ldots,P, \tag{1.6}$$

where

$$a_{ik} = \mathbb{E}[c(Z)\Phi_i(Z)\Phi_k(Z)],\ f_k(x,t) = E[f(x,t,Z)\Phi_k(Z)] \quad 0 \le i,k \le P. \tag{1.7}$$

We have obtained a coupled system of $(P+1)\times(P+1)$ deterministic transport equations, whose notation can be simplified into a matrix form, i.e.

$$\frac{d\mathbf{v}(x,t)}{dt} = \mathbf{A}\frac{d\mathbf{v}(x,t)}{dx} + \mathbf{f}, \tag{1.8}$$

where $\mathbf{v} = (v_0, v_1, \ldots, v_P)^T$, $\mathbf{f} = (f_0, f_1, \ldots, f_P)$ and $\mathbf{A}$ is $(P+1)\times(P+1)$ matrix whose entries are a_{ik}. Note, by definition $a_{ik} = a_{ki}$, so that $\mathbf{A}$ is a symmetric matrix, i.e. $\mathbf{A}^T = \mathbf{A}$.

Theorem 1.1. *Consider the deterministic system* (1.8) *where the coefficients are defined in* (1.7). *Then if* $C(z) \ge 0$ *(resp.* $C(z) \le 0$*) for all* $z \in \mathbb{R}$*, then the eigenvalues of* $\boldsymbol{A}$ *are all non-negative (resp. non-positive); if* $C(z)$ *changes sign, then* $\boldsymbol{A}$ *has both positive and negative eigenvalues for sufficiently large* P.

Therefore, the obtained system (1.8) is symmetric, so there exists an orthogonal matrix $\mathbf{Q}$ ($\mathbf{Q}^T = \mathbf{Q}^{-1}$), such that $\mathbf{Q}^T\mathbf{A}\mathbf{Q} = \Lambda$ or equivalently $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$, where Λ is diagonal matrix with eigenvalues of $\mathbf{A}$, i.e.,

$$\Lambda = diag(\lambda_0, \ldots, \lambda_{j_+}, \ldots, \lambda_{j_-}, \ldots, \lambda_N).$$

Entries $\lambda_0, \ldots \lambda_{j_-}$ are booked for negative eigenvalues and $\lambda_{j_+}, \ldots, \lambda_N$ are for positive, the rest, if they exist, are for zeros. If we multiply (1.8) with $\mathbf{Q}^T$ by left side, and denote $\mathbf{r} = (r_0, r_1, \ldots, r_P) = \mathbf{Q}^T\mathbf{v}$, $\hat{\mathbf{f}} = (\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_P) = \mathbf{Q}^T\mathbf{f}$, we obtain following diagonal system

$$\frac{d\mathbf{r}(x,t)}{dt} = \Lambda\frac{d\mathbf{r}(x,t)}{dx} + \hat{\mathbf{f}}, \tag{1.9}$$

with initial condition,

$$\mathbf{r}(x,0) = \mathbf{Q}^T v(x,0). \tag{1.10}$$

In order to impose boundary conditions to diagonal deterministic system (1.9)-(1.10) we ought to use sign of eigenvalues, i.e.,

$$\begin{aligned} r_j(1,t) &= \sum_{k=0}^{P} q_{kj} v_k(1,t), \qquad j = 0, \dots, j_-, \\ r_j(-1,t) &= \sum_{k=0}^{P} q_{kj} v_k(-1,t), \qquad j = j_+, \dots, P, \end{aligned} \tag{1.11}$$

where q_{jk} are entries of matrix $\mathbf{Q}$. Coefficients $v_k(1,t)$ are obtained using gPC expansion of functions $u_+(t,Z)$, and $u_-(t,Z)$,

$$\begin{aligned} u_+(t,Z) &= \sum_{k=0}^{P} v_k(1,t)\Phi_k(Z), \\ u_-(t,Z) &= \sum_{k=0}^{P} v_k(-1,t)\Phi_k(Z). \end{aligned} \tag{1.12}$$

2. Numerical simulations

Example 2.1. Now, we present some numerical examples to support the elaborated theory. We are going to use different random variables, as well as initial and boundary conditions. First, we consider

$$\begin{aligned} \frac{du(x,t,Z)}{dt} &= Z\frac{du(x,t,Z)}{dx}, \qquad -1 < x < 1,\ t \geq 0, \\ u(x,0,Z) &= \cos x, \qquad -1 < x < 1, \end{aligned} \tag{2.1}$$

where $Z \sim \mathcal{U}(-1,1)$ is uniformly distributed random variable, with PDF $\rho(z) = 1/2$ and suitable Legendre polynomials. The exact solution is

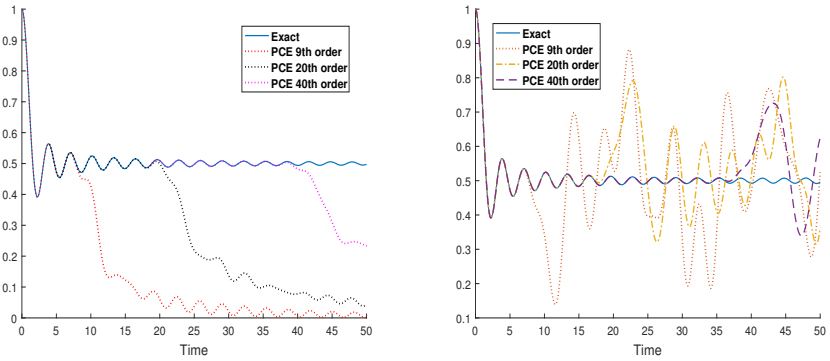
$$u_{exact} = \cos(x - Zt).$$

We suppose that solution of (2.1) has form

$$u_{approx}(x,t,Z) = \sum_{k=0}^{N} u_k(x,t)\Phi_k(Z).$$

There are two ways to obtain coefficients $u_k(x,t)$, namely, first one is to use exact solution u_{exact} where in every step we have to find expectation $u_k(x,t) = E[u_{exact}\Phi_k]$. Second way is to use described gPC Galerkin method and obtain system of deterministic equations, whose solutions are $u_k(x,t)$.
Figure 1, shows these mentioned methods. Namely, on both graphs we see that for fixed $x = x_0$, obtained approximations are accurate up to some point in time. Obviously, it depends on height of gPC order which is employed. As the order of polynomial chaos increases, accuracy in time is longer and better. Nevertheless, graphs are not the same, on the left one we see that after losing accuracy solutions converge to zero, whereas, on the right one, solutions continue to diverge. Obviously, the first method is inapplicable when the exact solution is unknown.
Accuracy will be measured by the mean-square error

$$e_m(N,t) = \max_x (E[(u_{exact} - u_{approx})^2])^{1/2},$$



(a) Direct method (b) Galerkin method

Figure 1: Solution of PDE (2.1) for $x = 0$, with $Z \sim \mathcal{U}(-1, 1)$

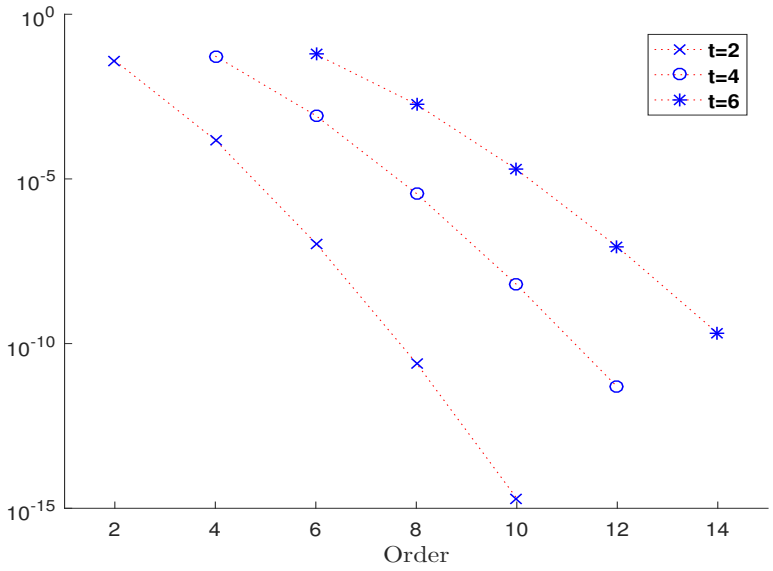
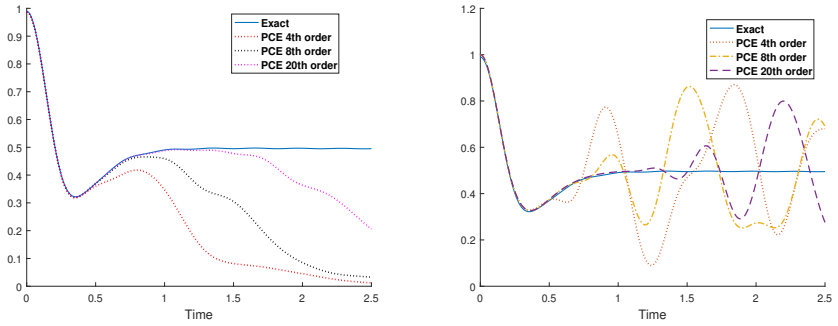


Figure 2: Mean-square error for $Z \sim \mathcal{U}(-1, 1)$ and different t

and results which are presented in Figure 2, shows exponential convergence of error in each time moment.

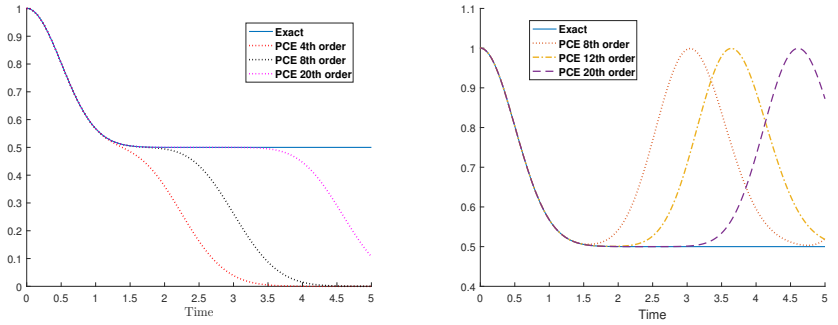
Also, we consider equation (2.1) for gamma and normal distributions. The obtained numerical results are presented in Figures 3 and 4. Conclusions for these two distributions are same as in previous case, but we notice that higher orders are required for the same length in time as were needed for uniform distribution.

Example 2.2. Now, let us consider following example with boundary conditions,



(a) Direct method (b) Galerkin method

Figure 3: Solution of PDE (2.1), for $x = 0$, with $Z \sim \Gamma(3, 1)$



(a) Direct method (b) Galerkin method

Figure 4: Solution of PDE (2.1), for $x = 0$, with $Z \sim \mathcal{N}(0, 1)$

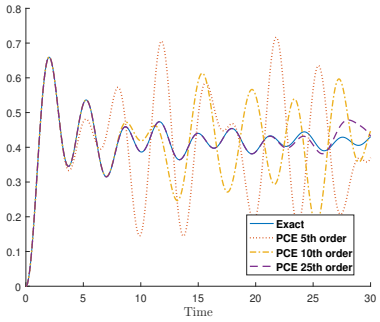
and discontinuity in random space.

$$
\begin{aligned}
u_t(x,t,Z) &= C(Z)u_x(x,t,Z), && -1 \le x \le 1,\ t > 0 \\
u(x,0,Z) &= kZ\sin(kx), && -1 \le x \le 1,\ Z > 0 \\
u(x,0,Z) &= 2kZ\sin(2kx), && -1 \le x \le 1,\ Z < 0.
\end{aligned} \tag{2.2}
$$

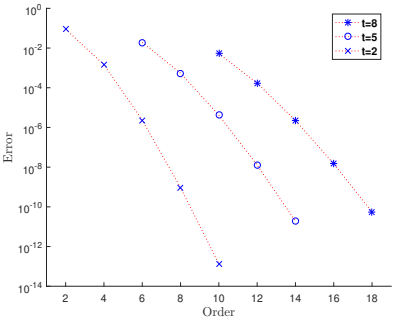
Here, $C(Z) = \sigma Z$, where $0 < \sigma < 1$, controlling the variability of random input and $k > 0$ is a real constant. We give boundary conditions as

$$
\begin{aligned}
u(1,t,Z) &= kZ\sin(k(1 + C(Z)t)), && Z > 0, \\
u(-1,t,Z) &= 2kZ\sin(2k(-1 + C(Z)t)), && Z < 0.
\end{aligned} \tag{2.3}
$$

Exact solution of (2.2)-(2.3) is $u_e = \sin(k(x + C(Z)t))$ for $Z > 0$, and $u_e = \sin(2k(x + C(Z)t))$ for $Z < 0$. The numerical solutions are solved with $\sigma = 0.5$ and $k = 1$. The boundary conditions are implemented via the eigenvalue analysis explained in theoretical part of section. We can see exponential convergence of the mean-square error from 10^0 to 10^{-12} in only few steps, in Figure 5b. Obviously, discontinuity in random space is not a problem for Galerkin method, because we obtain continuous functions in spatial and time domain.



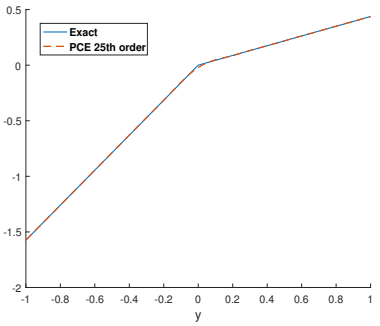
(a) Solutions for different orders of PCE



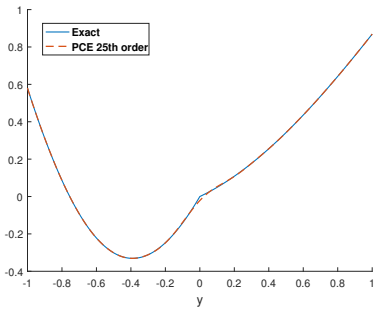
(b) Error for different values of t

Figure 5: Numerical solution of (2.2)-(2.3), for $x = 0$, with $Z \sim \mathcal{U}(-1, 1)$

Nevertheless, we notice in Figure 6 that the exact solution has some kind of discontinuity around point $y = 0$, where approximation has oscillations which are consequence of Gibbs phenomena. Problem (2.2)-(2.3) can be solved by taking other



(a)



(b)

Figure 6: Approximation of $u(x, t, Z)$ for $x = 0.453$ and (a) $t = 0$ and (b) $t = 1.2$

types of distribution for random input Z. Such an extension is more-less straightforward, because all we need is to use suitable type of gPC basis of polynomials, and follow the described steps.

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