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EXTENDED GEVREY REGULARITY WITH APPLICATION TO WAVELETS[1]

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Abstract. In this paper, we investigate several fundamental properties of the extended Gevrey classes. Furthermore, as an application, we construct wavelets whose regularity is characterized by these classes.

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1. Introduction

We will give a short review on some of the properties of the extended Gevrey classes. Moreover, we will use the following function

$$\begin{aligned} &f_{\rho,\sigma}(t) = e^{-\rho g_\sigma\left(\frac{1}{t}\right)}, \quad \rho > 0,\ \sigma > 1, \\ &\text{where} \\ &g_\sigma(t) = \frac{\ln^{\frac{\sigma}{\sigma-1}}(1+|t|)}{W^{\frac{1}{\sigma-1}}(\ln(1+|t|))}, \quad t \in \mathbb{R}, \end{aligned} \tag{1.1}$$

and W denotes the principal branch of *Lambert W function*, to construct a wavelet which is extended Gevrey regular in both time and frequency domains.

Extended Gevrey classes were introduced in [10] to describe regularity between the Gevrey classes and C^∞. The derivatives of their elements are controlled by sequences of *rapid growth*

$$\{M_p^{\tau,\sigma} = p^{\tau p^\sigma}\}_{p\in\mathbb{N}_0}, \quad \tau > 0,\ \sigma > 1. \tag{1.2}$$

These classes provide an important example of *weight matrix classes* of ultradifferentiable functions (see [5, 13]). Moreover, the function in (1.1) can be estimated in terms of the associated function corresponding to the sequences

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in (1.2). For some applications of the theory of extended Gevrey classes, we refer to [1, 2, 6, 7].

Lambert W function (see [9]) is defined as the multivalued inverse of the function $z \mapsto ze^z$ for $z \in \mathbb{C}$. It splits the complex plane into infinitely many regions. Throughout the paper we will use the notation $W(z)$, $z \in \mathbb{C} \setminus (-\infty, -e^{-1}]$, for its principal branch (instead of common $W_0(z)$). The boundary curve of its principal branch is given by:

$$\{(-t\operatorname{ctg}t, t) \in \mathbb{R}^2 \mid -\pi < t < \pi\}.$$

2. Extended Gevrey classes

We start with the following Lemma that contains the properties of sequences given in (1.2).

Lemma 2.1. *Let $\tau > 0$, $\sigma > 1$, $M_0^{\tau,\sigma} = 1$, and $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$, $p \in \mathbb{N}$. Then there exists constant $C > 1$ such that:*

$(M.1)$ $(M_p^{\tau,\sigma})^2 \leq M_{p-1}^{\tau,\sigma} M_{p+1}^{\tau,\sigma}$, $p \in \mathbb{N}$,

$\widetilde{(M.2)}$ $M_{p+q}^{\tau,\sigma} \leq C^{p^\sigma + q^\sigma} M_p^{\tau 2^{\sigma-1},\sigma} M_q^{\tau 2^{\sigma-1},\sigma}$, $p, q \in \mathbb{N}_0$,

$\widetilde{(M.2)}'$ $M_{p+1}^{\tau,\sigma} \leq C^{p^\sigma} M_p^{\tau,\sigma}$, $p \in \mathbb{N}_0$,

$(M.3)'$ $\displaystyle\sum_{p=1}^{\infty} \frac{M_{p-1}^{\tau,\sigma}}{M_p^{\tau,\sigma}} < \infty.$

In addition, if $\sigma_2 > \sigma_1 > 1$ and $\tau_0 > 0$ then for every $h, \tau > 0$ there exists $C > 0$ such that

$$h^{p^{\sigma_1}} M_p^{\tau_0,\sigma_1} \leq C M_p^{\tau,\sigma_2}$$

Let us briefly discuss the borderline case $\sigma = 1$. Note that $M_p := M_p^{\tau,1} = p^{\tau p}$, $\tau > 0$, is the Gevrey sequence, and the conditions $(M.1)$, $\widetilde{(M.2)}'$, and $\widetilde{(M.2)}$ are classical Komatsu conditions $(M.1)$, $(M.2)'$, and $(M.2)$, respectively. Moreover, M_p satisfies condition $(M.3)'$ if and only if $\tau > 1$, which ensures that the corresponding ultradifferentiable classes contain functions with compact support (see [8]). The proof of Lemma 2.1 is given in [10].

Next we define extended Gevrey classes $\mathcal{E}_{\tau,\sigma}(\mathbb{R})$.

Definition 2.1. *Let $\tau > 0$ and $\sigma > 1$ and $M_p^{\tau,\sigma} = p^{\tau p^\sigma}$ for $p \in \mathbb{N}$, $M_0^{\tau,\sigma} = 1$. A smooth function ϕ belongs to $\mathcal{E}_{\tau,\sigma}(\mathbb{R})$ if*

$$(\forall\, \mathbb{K} \subset\subset \mathbb{R})(\exists C > 0) \quad \sup_{x \in \mathbb{K}} |\phi^{(p)}(x)| \leq C^{p^\sigma + 1} M_p^{\tau,\sigma},\ p \in \mathbb{N}_0.$$

We denote

$$\mathcal{E}_\sigma(\mathbb{R}) = \bigcup_{\tau > 0} \mathcal{E}_{\tau,\sigma}(\mathbb{R}).$$

Again for $\sigma = 1$ we obtain some of the well-known classes. For instance that $\mathcal{A}(\mathbb{R}) := \mathcal{E}_{1,1}(\mathbb{R})$, and $\mathcal{G}_t(\mathbb{R}) := \mathcal{E}_{1,t}(\mathbb{R})$ for $t > 1$; these are the spaces of locally analytic functions and Gevrey functions of order t, respectively.

In the following Proposition we collect some of the basic properties of the extended Gevrey classes. The proof relies on the properties of $M_p^{\tau,\sigma}$ given in Lemma 2.1 (see [10, 11, 12, 13]).

Proposition 2.1. *Let $\tau > 0$ and $\sigma > 1$. Then*

i) For $\sigma_2 > \sigma_1 > 1$ we have

$$\mathcal{A}(\mathbb{R}) \subset \bigcup_{t>1} \mathcal{G}_t(\mathbb{R}) \subset \bigcap_{\tau>0} \mathcal{E}_{\tau,\sigma_1}(\mathbb{R}) \subset \mathcal{E}_{\tau,\sigma_1}(\mathbb{R}) \subset \mathcal{E}_{\sigma_1} \subset \mathcal{E}_{\sigma_2} \subset C^\infty(\mathbb{R}).$$

ii) $\mathcal{E}_{\tau,\sigma}(\mathbb{R})$ is closed under the pointwise multiplication.

iii) $\mathcal{E}_{\tau,\sigma}(\mathbb{R})$ is closed under finite order derivation.

iv) $\mathcal{E}_{\tau,\sigma}(\mathbb{R})$ is closed under superposition. In particular, if $F(x) \in \mathcal{A}(\mathbb{R})$ and $f(x) \in \mathcal{E}_{\tau,\sigma}(\mathbb{R})$ then $F(f(x)) \in \mathcal{E}_{\tau,\sigma}(\mathbb{R})$.

v) $\mathcal{E}_{\tau,\sigma}(\mathbb{R})$ is invariant under translations and dilatations.

Let us define a function associated to $M_p^{\tau,\sigma}$, $p \in \mathbb{N}_0$.

Definition 2.2. *Let $\tau > 0$ and $\sigma > 1$. Then the associated function to the sequence $M_p^{\tau,\sigma}$ is given by*

$$T_{\tau,\sigma}(k) = \sup_{p\in\mathbb{N}_0} \ln \frac{k^p}{M_p^{\tau,\sigma}}, \qquad k > 0. \tag{2.1}$$

In the following Theorem we estimate $T_{\tau,\sigma}(k)$ in terms of the Lambert W function. The proof can be found in [13], and for a more general version we refer to [12].

Theorem 2.1. *For $\tau > 0$ and $\sigma > 1$ we have*

$$T_{\tau,\sigma}(k) \asymp \tau^{-\frac{1}{\sigma-1}} \frac{\ln^{\frac{\sigma}{\sigma-1}}(1+k)}{W^{\frac{1}{\sigma-1}}(\ln(1+k))}, \quad k \geq 0.$$

In particular, $T_{\tau,\sigma}(|k|) \asymp \rho\, g_\sigma(k)$, $k \in \mathbb{R}$, for $\rho = \tau^{-\frac{1}{\sigma-1}}$ where g_σ is given in (1.1).

Next we discuss the connection between functions given in (1.1), sequences in (1.2) and associated function (2.1).

Let us define

$$h_{\tau,\sigma}(t) := e^{-T_{\tau,\sigma}(1/t)} = \inf_{p\in\mathbb{N}_0} M_p^{\tau,\sigma} t^p, \quad t \geq 0.$$

Then Theorem 2.1 implies that for any $\rho > 0$ and $\sigma > 1$ there exist constants $A, B, \tau_1, \tau_2 > 0$ such that

$$Ah_{\tau_1,\sigma}(|t|) \leq f_{\rho,\sigma}(t) \leq Bh_{\tau_2,\sigma}(|t|), \quad t \in \mathbb{R}, \tag{2.2}$$

where $f_{\rho,\sigma}$ is given in (1.1). Note that the right-hand side of (2.2) implies that for any given $\rho > 0$ there exists $B_1, \tau > 0$ such that

$$\sup_{t>0} \frac{f_{\rho,\sigma}(t)}{t^p} \leq B_1 M_p^{\tau,\sigma}, \quad p \in \mathbb{N}_0. \tag{2.3}$$

Estimate (2.3) (together with Caushy's integral formula) can be used to prove that

$$f_{\rho_\sigma,\sigma}(t) = \exp\left\{-e^{-\frac{1}{\sigma}} W^{-\frac{1}{\sigma-1}}\left(\ln\left(1+\frac{1}{|t|}\right)\right) \ln^{\frac{\sigma}{\sigma-1}}\left(1+\frac{1}{|t|}\right)\right\}, \; t \in \mathbb{R}, \tag{2.4}$$

is an example of the extended Gevrey function. In particular, following Proposition holds.

Proposition 2.2. *If $\sigma > 1$ and the function $f_{\rho_\sigma,\sigma}$ is given in* (2.4) *then:*

a) $\lim_{t\to 0} f^{(j)}_{\rho_\sigma,\sigma}(t) = 0$, $j \in \mathbb{N}$,

b) $f_{\rho_\sigma,\sigma}(t) \in \mathcal{E}_\sigma(\mathbb{R})$.

We conclude this section with the Paley–Wiener theorem, which connects the time-domain regularity of a function with its decay rate in the Fourier domain. The decay rate we use is given in the following definition.

Definition 2.3. *Let $\sigma > 1$ and $g_\sigma(\xi) = \dfrac{\ln^{\frac{\sigma}{\sigma-1}}(1+|\xi|)}{W^{\frac{1}{\sigma-1}}(\ln(1+|\xi|))}$ be as in* (1.1). *A continuous function $f(\xi)$ belongs to $\mathbf{\Gamma}_\sigma(\mathbb{R})$, if there exist $\rho > 0$ and $C > 0$ such that*

$$|f(\xi)| \leq C\, e^{-\rho g_\sigma(\xi)}, \quad \xi \in \mathbb{R}.$$

We now state the Paley–Wiener theorem. For the proof we refer to [12].

Theorem 2.2. *Let $\sigma > 1$.*

a) Let φ be a C^∞ function with compact support. If $\varphi(x) \in \mathcal{E}_\sigma(\mathbb{R})$ then $\widehat{\varphi}(\xi) \in \mathbf{\Gamma}_\sigma(\mathbb{R})$.

b) If the function φ is such that $\widehat{\varphi}(\xi) \in \mathbf{\Gamma}_\sigma(\mathbb{R})$ then $\varphi(x) \in \mathcal{E}_\sigma(\mathbb{R})$.

3. Wavelet construction

In this section, we use the functions given in (1.1) to construct an orthonormal wavelet ψ such that both ψ and its Fourier transform $\hat{\psi}$ belong to $\mathcal{E}_\sigma(\mathbb{R})$. Our construction follows the ideas of multiresolution analysis (MRA for short, see [3, 4]). We refer to [14] for the construction of band-limited wavelets, that is, wavelets whose Fourier transform has compact support.

Let $m_0(\xi)$ be a *low-pass filter* of the orthonormal wavelet that satisfies following minimal requirements:

i) m_0 is a continuous, even, $2\pi-$ periodic function on $\mathbb{R}, m_0(0) = 1$,

ii) $\inf\limits_{\xi\in[-\pi/3,\pi/3]} |m_0(\xi)| \neq 0$,

iii) $|m_0(\xi)|^2 + |m_0(\xi+\pi)|^2 = 1$.

Then the Fourier transform of the *scaling function* φ is continuous and

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \quad \xi \in \mathbb{R}.$$

Moreover, if m_0 is smooth then $\hat{\varphi}$ is smooth.

Fourier transform of the orthonormal wavelet is then given by

$$\hat{\psi}(\xi) = e^{i\frac{\xi}{2}} \overline{m_0\Big(\frac{\xi}{2}+\pi\Big)} \hat{\varphi}\Big(\frac{\xi}{2}\Big), \quad \xi \in \mathbb{R}. \tag{3.1}$$

In particular, it is sufficient to construct a low-pass filter that belongs to $\mathcal{E}_\sigma$. The properties of extended Gevrey classes given in Proposition 2.1, together with the Paley–Wiener theorem (2.2), imply that the orthonormal wavelet defined in (3.1) satisfies $\psi, \hat{\psi} \in \mathcal{E}_\sigma(\mathbb{R})$.

We construct the desired low-pass filter as follows. Let

$$\gamma_\sigma(\eta) := \begin{cases} f_\sigma(\eta) & ,\eta > 0 \\ 0 & ,\eta \leq 0, \end{cases}$$

where $f_\sigma = f_{1,\sigma}$ is given in (1.1).

For

$$\delta_\sigma(\xi) := \begin{cases} \Big(\int_0^1 \gamma_\sigma(\eta)\gamma_\sigma(1-\eta)\,d\eta\Big)^{-1} \int_0^\xi \gamma_\sigma(\eta)\gamma_\sigma(1-\eta)\,d\eta & ,\xi > 0 \\ 0 & ,\xi \leq 0, \end{cases}$$

Let us define

$$\theta_\sigma(\xi) := \left(1 - \delta_\sigma\left(\frac{5\xi-\pi}{3\pi}\right)\right)\delta_\sigma\left(\frac{|\xi - \frac{2\pi}{3}|}{d}\right), \quad \xi \in \left[\frac{\pi}{2}, \pi\right), d \in \left(0, \frac{\pi}{6}\right].$$

Note that θ_σ can be extended to $[0, 2\pi]$ by $\theta_\sigma(\xi) + \theta_\sigma(\xi+\pi) = 1$, and then further extended on $\mathbb{R}$ so that it is 2π periodic.

Finally, low-pass filter $m_0 \in \mathcal{E}_\sigma$ is given by

$$m_0(\xi) = \sin\left(\frac{\pi}{2}\theta_\sigma(\xi)\right), \ \xi \in \mathbb{R}.$$

The further details concerning wavelet construction can be found in [15].

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