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A NEW FUZZY CONTRACTIVE CONDITION IN FUZZY METRIC SPACES WITH APPLICA-TIONS TO IMAGE PROCESSING

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Abstract. We introduce a new fuzzy contractive condition in fuzzy metric spaces equipped with continuous t-norms of H-type and establish a fixed point theorem under this assumption. A concrete example is provided, together with an application to iterative image filtering.

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1. Introduction

Fixed point theory has been a cornerstone of modern mathematical analysis since Banach introduced his contraction principle in 1922, guaranteeing the existence and uniqueness of fixed points for self-mappings in complete metric spaces. Over time, numerous generalizations were developed to extend these results to more abstract settings, including fuzzy metric spaces, which allow for modeling uncertainty and imprecision encountered in real-world problems. Fixed point theory in fuzzy metric spaces has been extensively studied, providing the foundation for our work [1, 2, 4, 5].

Fuzzy sets, introduced by Zadeh in 1965 [13], provide a mathematical framework for dealing with uncertainty. Fuzzy metric spaces, formalized by Kramosil and Michalek (1975) and further developed by George and Veeramani (1994), incorporate a continuous t-norm to define a generalized notion of distance. These spaces have found applications in image processing, decision making, control theory, and other fields where uncertainty plays a crucial role.

2. Preliminaries

Definition 2.1. [8] A binary operation $* : [0,1]^2 \to [0,1]$ is a t-norm if it is commutative, associative, nondecreasing in each argument, and has 1 as the neutral element ($a * 1 = a$).

Example 2.2. Basic examples of t-norms: Minimum t-norm: $*_M(a,b) = \min(a,b)$, $*_P(a,b) = a \cdot b$, $*_L(a,b) = \max\{0, a+b-1\}$.

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Definition 2.3. [7] Let $*$ be a t-norm and let the sequence of functions $*_n : [0,1] \to [0,1]$ $(n \in \mathbb{N})$ be defined as follows:

$$*_1(x) = *(x,x), \qquad *_{n+1}(x) = *(*_n(x), x), \quad (n \in \mathbb{N},\, x \in [0,1]).$$

A t-norm $*$ is said to be of *Hadžić type* (or *H-type*) if the family $\{*_n(x)\}_{n\in\mathbb{N}}$ is equicontinuous at the point $x = 1$; that is, for every $\lambda \in (0,1)$ there exists $\delta(\lambda) \in (0,1)$ such that the implication

$$x > 1 - \delta(\lambda) \;\Rightarrow\; *_n(x) > 1 - \lambda, \qquad \text{for all } n \in \mathbb{N},$$

holds true.

Definition 2.4. [5] Let X be a nonempty set and $*$ a continuous t-norm. A function

$$M : X \times X \times (0,\infty) \to [0,1]$$

is called a fuzzy metric if for all $x, y, z \in X$ and $s, t > 0$, the following hold:

1. $M(x,y,t) = 1$ if and only if $x = y$,
2. $M(x,y,t) = M(y,x,t)$,
3. $M(x,y,t) * M(y,z,s) \le M(x,z,t+s)$,
4. $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous,

The triple $(X, M, *)$ is called a fuzzy metric space

Definition 2.5. [6] $M(x,\, y,\, \cdot)$ is nondecreasing for all $x,\, y \in X$.

Definition 2.6. [5] Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\} \subset X$ is said to converge to $x \in X$ if for every $\varepsilon \in (0,1)$ and $t > 0$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $M(x_n, x, t) > 1 - \varepsilon$.

Definition 2.7. [5] A sequence $\{x_n\} \subset X$ is called Cauchy sequence if for every $\varepsilon \in (0,1)$ and $t > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \ge N$, $M(x_n, x_m, t) > 1 - \varepsilon$.

Definition 2.8. [1] A mapping $T : X \to X$ is a Banach contraction if there exists $k \in (0,1)$ such that

$$M(Tx, Ty, t) \ge M(x, y, t/k), \quad \forall x, y \in X, t > 0. \tag{2.1}$$

3. A New Fuzzy Contractive Condition

Definition 3.1. Let $(X, M, *)$ be a fuzzy metric space and $T : X \to X$ a mapping. We say that T satisfies the new fuzzy contractive condition with respect to a strictly decreasing continuous function $\phi : [0,1] \to [0,1]$, with $\phi(1) = 0$ and $\phi(r) < r$ for $r \in [0,1)$, if for all $x, y \in X$ and $t > 0$:

(3.1)

$$\phi(M(Tx,Ty,t)) \le \max\Big\{\phi(M(x,y,t)),\ \phi(M(x,Tx,t)) * \phi(M(y,Ty,t))\Big\}.$$

Remark 3.2. Clearly, if condition (2.1) holds, then condition (3.1) is also satisfied.

Theorem 3.3. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ satisfy the new fuzzy contractive condition with function ϕ as above. Then T has a unique fixed point $x^* \in X$, and for any $x_0 \in X$, the sequence $x_{n+1} = Tx_n$ converges to x^*.*

Proof. Let $x_0 \in X$ and define $x_{n+1} = Tx_n$ for $n \geq 0$. Define

$$a_n(t) = \phi(M(x_n, x_{n-1}, t)), \quad b_n(t) = \phi(M(x_n, x_{n+1}, t)).$$

From the contractive condition, we have

$$b_n(t) = \phi(M(Tx_n, Tx_{n-1}, t)) \leq \max\{a_n(t), a_n(t) * b_n(t)\} = a_n(t),$$

since $a_n(t) * b_n(t) \leq a_n(t)$ for $a_n(t), b_n(t) \in [0, 1]$. Since ϕ is strictly decreasing, the inequality

$$\phi(M(x_{n+1}, x_n, t)) \leq \phi(M(x_n, x_{n-1}, t)) \text{ implies } M(x_{n+1}, x_n, t) \geq M(x_n, x_{n-1}, t),$$

so the sequence $\{M(x_{n+1}, x_n, t)\}$ is monotone increasing bounded from above with 1 so its convergent to some $a(t)$. Suppose that $a(t) < 1$. Suppose

$$a(t) = \lim_{n\to\infty} M(x_{n+1}, x_n, t) < 1 \implies \phi(a(t)) < a(t) < 1,$$

which leads to a contradiction. Therefore, it is necessary that

$$\lim_{n\to\infty} M(x_{n+1}, x_n, t) = 1.$$

Let $\varepsilon > 0$ be given. Because the sequence $\{M(x_n, x_{n+1}, t)\}$ converges to 1, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$M(x_n, x_{n+1}, t) > 1 - \varepsilon, \quad t > 0.$$

Now, using the recursive property of the fuzzy metric and the t-norm of H-type, for any $m > n \geq N$, we have

$$M(x_n, x_m, t) \geq M\Big(x_n, x_{n+1}, \frac{t}{m-n}\Big) * \cdots * M\Big(x_{m-1}, x_m, \frac{t}{m-n}\Big),$$

and by the choice of N and the equicontinuity of $*$ at 1, this $*$-combination remains arbitrarily close to 1, which implies

$$M(x_n, x_m, t) \geq 1 - \varepsilon$$

for all $m, n \geq N$. Hence, $\{x_n\}$ is a Cauchy sequence in the fuzzy metric space $(X, M, *)$.

To show x^* is a fixed point, note

$$\begin{aligned} & \phi(M(Tx^*, x^*, t)) = \phi(M(Tx^*, Tx_n, t)) \\ \leq \; & \max\{\phi(M(x^*, x_n, t)), \phi(M(x^*, Tx^*, t)) * \phi(M(x_n, Tx_n, t))\}. \end{aligned}$$

Letting $n \to \infty$ yields $\phi(M(Tx^*, x^*, t)) \leq \phi(1)$, and by the strict monotonicity of ϕ, we obtain $Tx^* = x^*$. Uniqueness follows similarly from the contractive condition. □

Example 3.4. Let $X = [\frac{1}{2}, 1] \subset \mathbb{R}$ with the fuzzy metric

$$M(x, y, t) = \frac{t}{t + |x - y|}, \quad t > 0,$$

and the minimum t-norm $a * b = \min\{a, b\}$. Define

$$\phi(r) = r - r^2, \quad r \in [0, 1].$$

Then ϕ is continuous on $[0, 1]$, $\phi(1) = 0$, $\phi(r) < r$ for all $r \in [0, 1)$, and ϕ is strictly decreasing on $[1/2, 1]$, since $\phi'(r) = 1 - 2r < 0$ for $r > 1/2$.

Define the operator $T : X \to X$ by

$$T(x) = \frac{x}{2} + \frac{1}{4}.$$

Clearly, T maps X into itself, since for $x \in [\frac{1}{2}, 1]$, we have $T(x) \in [\frac{1}{2}, 1]$.

Verification of the new fuzzy contractive condition.

For any $x, y \in X$ and $t > 0$, we have

$$|Tx - Ty| = \frac{|x - y|}{2},$$

and therefore

$$M(Tx, Ty, t) = \frac{t}{t + |x - y|/2}.$$

Since $M(Tx, Ty, t) \geq M(x, y, t)$ and ϕ is strictly decreasing on $[1/2, 1]$, it follows that

$$\phi(M(Tx, Ty, t)) \leq \phi(M(x, y, t)).$$

Consequently,

$$\phi(M(Tx, Ty, t)) \leq \max\Big\{\phi(M(x, y, t)),\, \phi(M(x, Tx, t)) * \phi(M(y, Ty, t))\Big\}.$$

Thus, the new fuzzy contractive condition (3.1) is satisfied and $x = \frac{1}{2}$ is a fixed point.

4. Application: Iterative Filtering in Image Processing

Iterative filtering consists of repeatedly applying an image operator T to an initial image I_0, generating the sequence $I_{n+1} = T(I_n)$ until it stabilizes at a limit image I^*. Applications of fuzzy metrics and iterative filtering in image processing have been extensively studied in the literature, demonstrating both theoretical importance and practical efficiency [3, 4, 9, 10, 11, 12].

Consider a small grayscale image patch I_0 corrupted by impulse noise:

$$I_0 = \begin{bmatrix} 120 & 125 & 130 \\ 122 & 255 & 128 \\ 121 & 123 & 127 \end{bmatrix}.$$

Define T as a continuous approximation of a median filter:

$$T(I)(p) = \frac{1}{|N(p)|} \sum_{q \in N(p)} I(q).$$

The fuzzy metric and the contractive function are:

$$M(I(p), J(p), t) = \frac{t}{t + |I(p) - J(p)|}, \quad \phi(r) = r - r^2, \quad * = \min.$$

Starting with I_0, define the sequence $I_{n+1} = T(I_n)$. The iterations produce:

$$I_1 \approx \begin{bmatrix} 123 & 136 & 133 \\ 142 & 136 & 136 \\ 129 & 133 & 127 \end{bmatrix} \quad I_2 \approx \begin{bmatrix} 134 & 135 & 134 \\ 135 & 134 & 134 \\ 133 & 134 & 133 \end{bmatrix} \quad I_3 \approx \begin{bmatrix} 134 & 134 & 134 \\ 134 & 134 & 134 \\ 134 & 134 & 134 \end{bmatrix}.$$

The sequence converges to the unique fixed point I^*.

To verify that the operator T is fuzzy contractive, we compare pixel values in two consecutive iterations. For any pixel p and $t > 0$,

$$M(I_{n+1}(p), I_n(p), t) = \frac{t}{t + |\, I_{n+1}(p) - I_n(p)\,|}.$$

Applying $\phi(r) = r - r^2$, we obtain

$$\phi(M(I_{n+1}(p), I_n(p), t)) = M(I_{n+1}(p), I_n(p), t) - M(I_{n+1}(p), I_n(p), t)^2.$$

The contractive condition requires

$$\phi(M(I_{n+1}(p), I_n(p), t)) \leq \max\Big\{\phi(M(I_n(p), I_n(q), t)),$$
$$\phi(M(I_n(p), I_{n+1}(p), t)) * \phi(M(I_n(q), I_{n+1}(q), t))\Big\},$$

with $q \in N(p)$. Since T is an averaging operator and the t-norm is of H-type, the product term remains small, so the inequality holds at each pixel. Consequently, each iteration decreases the fuzzy distance between successive images, and the sequence $\{I_n\}$ converges to a unique stable image I^*.

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