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GENERAL ALGEBRAIC SOLUTION OF A FUZZY SYLVESTER MATRIX EQUATION [1]

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Abstract. In this paper, we propose a new algorithm based on {1}-inverses for finding all algebraic (strong) solutions of a fuzzy Sylvester matrix equation (FSME), $A\tilde{X} + \tilde{X}B = \tilde{C}$, where the coefficient matrices A and B are real matrices of orders $n \times n$ and $m \times m$, respectively, while $\tilde{C}$ is a given matrix of fuzzy numbers. The proposed method is illustrated through a numerical example.

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1. Introduction

In numerous practical applications involving systems of linear equations, the available data are often imprecise or uncertain. Consequently, there is a need to develop methods for solving systems of equations where certain parameters are represented as fuzzy numbers. A method for solving square fuzzy linear systems (FLS), $A\tilde{X} = \tilde{B}$, whose coefficient matrix A is real and $\tilde{B}$ is an arbitrary fuzzy number vector, was first introduced by Friedman et al. in [2]. In [5], Mihailović et al. presented the first algorithm for obtaining all solutions of rectangle FLS using the Moore-Penrose inverse of the coefficient matrix. In [1], Dragić et al. proposed the straightforward method for obtaining all algebraic (strong) solutions of an arbitrary dual fuzzy linear system (DFLS), $A\tilde{X} + \tilde{C} = B\tilde{X} + \tilde{D}$, where the coefficient matrices A and B are arbitrary real $m \times n$ matrices and $\tilde{C}$ and $\tilde{D}$ are given fuzzy number vectors.

In 2011, Salkuyeh [6] investigated the existence and uniqueness of a fuzzy solution to the fuzzy Sylvester matrix equation (FSME) of the form $A\tilde{X}+\tilde{X}B = \tilde{C}$, when matrices A and B are M-matrices. Salkuyeh transformed FSME into DFLS by using Kronecker products. Very recently, in [3], He et al. established an algorithm to obtain general strong fuzzy solutions to the FSME by using

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the BT inverse of the coefficient matrix, under certain additional conditions on the involved matrices, including their nonnegativity (or nonpositivity).

The main goal of this paper is to characterize the general algebraic (strong) solution of the FSME. Using the Kronecker product, we transform the given FSME into a dual fuzzy linear system, which is then extended to a classical system of linear equations, for which a necessary and sufficient condition for the existence of representative solutions (see page 4) is established. Furthermore, an algorithm based on the $\{1\}$-inverses is proposed to obtain the general algebraic solution, and its effectiveness is demonstrated through an example.

2. Preliminaries and previous research

Let us denote with $\mathcal{M}^{m\times n}$ the class of all $m \times n$ real matrices, $m, n \in \mathbb{N}$. For any $F \in \mathcal{M}^{m\times n}$, let a matrix $G \in \mathcal{M}^{n\times m}$ be such that it fulfills equation $FGF = F$. A such matrix G is called an $\{1\}$-inverse of F, and it will be denoted by $F^{(1)}$, and $F\{1\}$ denotes sets of all $\{1\}$-inverses of F.

Recall that a general solution of a classical linear system $AX = B$ is given in the next form:

$$X = A^{(1)}B + (I_n - A^{(1)}A)V, \quad A \in \mathcal{M}^{m\times n},\, B \in \mathcal{M}^{m\times 1}, \tag{2.1}$$

where $A^{(1)} \in A\{1\}$, $V \in \mathcal{M}^{n\times 1}$ is arbitrary, and I_n is identity matrix of size n.

A fuzzy number is a mapping $\tilde{u} : \mathbb{R} \to [0, 1]$ so that for each $\alpha \in (0, 1]$ the α-level set, $[\tilde{u}]^\alpha = \{x \in \mathbb{R} : \tilde{u}(x) \geq 0\}$, $\forall \alpha \in (0, 1]$, is a nonempty, closed and bounded interval, and support of $\tilde{u}$ defined as $supp(\tilde{u}) = \overline{\{x \in \mathbb{R} : \tilde{u}(x) > 0\}}$ is a nonempty, closed and bounded interval. The set of all fuzzy numbers is denoted by $\tilde{\mathcal{J}}$.

A fuzzy number $\tilde{u} \in \tilde{\mathcal{J}}$ is given in the parametric form $(\underline{u}, \overline{u})$, where the upper branch $\overline{u} : [0, 1] \to \mathbb{R}$ is non-increasing and left continuous, $\overline{u}(\alpha) = \sup[\tilde{u}]^\alpha$, whereas the lower branch $\underline{u} : [0, 1] \to \mathbb{R}$ is non-decreasing and left continuous, $\underline{u}(\alpha) = \inf[\tilde{u}]^\alpha$, such that $\underline{u}(\alpha) \leq \overline{u}(\alpha)$, for each $\alpha \in [0, 1]$. The addition, scalar multiplication and equality of fuzzy numbers are based on the interval arithmetic and the fact that α-cuts of a fuzzy number $\tilde{u}$, given by $[\tilde{u}]^\alpha = [\underline{u}(\alpha), \overline{u}(\alpha)]$, $\alpha \in [0, 1]$, are closed intervals (for more details, we refer the reader to [1, 2]).

Definition 2.1 ([4])**.** Let $P = [p_{ij}] \in \mathcal{M}^{m\times n}$, $p_i = (p_{1i}, p_{2i}, \ldots, p_{mi})^T$ be the i-th column of P, $i = 1, \ldots, n$, then the mn-dimensional vector $\mathrm{vec}(P) = V_P = \begin{bmatrix} p_1 & p_2 & \ldots & p_n \end{bmatrix}^T$ is called the extension on column of the matrix P.

Definition 2.2 ([4])**.** Let $P \in \mathcal{M}^{m\times n}$, $Q \in \mathcal{M}^{p\times q}$. Then the Kronecker product is defined by $P \otimes Q = \begin{bmatrix} p_{11}Q & \cdots & p_{1n}Q \\ \vdots & & \vdots \\ p_{m1}Q & \cdots & p_{mn}Q \end{bmatrix}$, where $P \otimes Q \in \mathcal{M}^{mp\times nq}$.
It holds $\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X)$.

Let $\tilde{X} = (\tilde{x}_1, \ldots, \tilde{x}_n)^T$ denotes a fuzzy number vector, where $\tilde{x}_i \in \tilde{\mathcal{J}}$, $i = 1, \ldots, n$. Let $\mathcal{V}_n$ denotes the class of all n-dimensional fuzzy number vectors.

Let $X = (\underline{x}_1, \ldots, \underline{x}_n, -\overline{x}_1, \ldots, -\overline{x}_n)^T$ denotes the associated $2n \times 1$ classical functional vector and let $\mathcal{F}$ denotes a class of all classical functional vectors. The vector $X \in \mathcal{F}$ is called the representative vector if all its components are adequate functions on the unit interval that represent the upper and lower branches of some fuzzy number vector $\tilde{X} \in \mathcal{V}_n$. A class of all representative classical functional vectors is denoted by $\mathcal{F}^{\mathcal{R}}$. For each $\tilde{X} \in \mathcal{V}_n$, the associated $n \times 1$ classical functional vectors are $\underline{X} = (\underline{x}_1, \ldots, \underline{x}_n)^T$ and $\overline{X} = (\overline{x}_1, \ldots, \overline{x}_n)^T$. The zero vector is denoted by O.

For each matrix $M \in \mathcal{M}^{m\times n}$, let $S_M \in \mathcal{M}^{2m\times 2n}$ be given by

$$S_M = \begin{bmatrix} M^+ & M^- \\ M^- & M^+ \end{bmatrix}, \tag{2.2}$$

where for each matrix $M \in \mathcal{M}^{m\times n}$ we denote $M^+ = \frac{1}{2}(M + |M|)$ and $M^- = \frac{1}{2}(|M| - M)$, $|M| = [|m_{ij}|]$.

Definition 2.3 ([3])**.** The fuzzy matrix equation

$$A\tilde{X} + \tilde{X}B = \tilde{C}, \tag{2.3}$$

where $A = [a_{ij}] \in \mathcal{M}^{n\times n}$, $B = [b_{ij}] \in \mathcal{M}^{m\times m}$, $\tilde{C} = [\tilde{c}_{ij}]$ is a $n \times m$ matrix of fuzzy numbers, and $\tilde{X}$ is a $n \times m$ unknown matrix of fuzzy numbers, is called a fuzzy Sylvester matrix equation (FSME).

A matrix of fuzzy numbers $\tilde{U}$, $\mathrm{vec}(\tilde{U}) = \tilde{V}_{\tilde{U}} \in \mathcal{V}_{2mn}$, $V_{\tilde{U}} \in \mathcal{F}^{\mathcal{R}}$, is a (strong) solution of (2.3) iff it holds:

$$A\tilde{U} + \tilde{U}B = \tilde{C}.$$

Theorem 2.4 ([3])**.** *Let $A \in \mathcal{M}^{n\times n}$, $B \in \mathcal{M}^{m\times m}$, $\tilde{X}$, and $\tilde{C}$ be $n \times m$ matrices of fuzzy numbers. Then, $A\tilde{X} + \tilde{X}B = \tilde{C}$ is equivalent to*

$$(I_m \otimes A)\tilde{V}_{\tilde{X}} + (B^T \otimes I_n)\tilde{V}_{\tilde{X}} = \tilde{V}_{\tilde{C}}, \tag{2.4}$$

where I_n and I_m denote the identity matrices of order n and m, respectively.

3. A new algorithm for solving FSME

Denote $P = I_m \otimes A$ and $Q = B^T \otimes I_n$. The matrix equation (2.4) can be extended to the following crisp form

$$\begin{bmatrix} P^+ & P^- \\ P^- & P^+ \end{bmatrix} \begin{bmatrix} V_{\tilde{X}} \\ \hline -\overline{V}_{\tilde{X}} \end{bmatrix} + \begin{bmatrix} Q^+ & Q^- \\ Q^- & Q^+ \end{bmatrix} \begin{bmatrix} V_{\tilde{X}} \\ \hline -\overline{V}_{\tilde{X}} \end{bmatrix} = \begin{bmatrix} V_{\tilde{C}} \\ \hline -\overline{V}_{\tilde{C}} \end{bmatrix}.$$

For each matrix $M \in \mathcal{M}^{m\times n}$, let $S_M \in \mathcal{M}^{2m\times 2n}$ be given by (2.2). Therefore, the matrix equation (2.4) can be extended into crisp systems of linear equations as follows:

$$(S_P + S_Q)V_{\tilde{X}}(\alpha) = V_{\tilde{C}}(\alpha),\ \alpha \in [0, 1], \tag{3.1}$$

(we write $(S_P + S_Q)V_{\tilde{X}} = V_{\tilde{C}}$) associated to the system (2.4). Denote the solution set of (3.1) with $\mathcal{ASS}$. Let us denote the subset of $\mathcal{ASS}$ which contains all representative solutions of (3.1) with $\mathcal{ASS}^{\mathcal{R}}$, i.e.,

$$\mathcal{ASS}^{\mathcal{R}}=\{V_{\tilde{X}} \in \mathcal{F}^{\mathcal{R}} |\ (S_P+S_Q)V_{\tilde{X}} = V_{\tilde{C}}\} \subseteq \{X \in \mathcal{F} |\ (S_P+S_Q)V_{\tilde{X}} = V_{\tilde{C}}\} = \mathcal{ASS}.$$

If $\mathcal{ASS}^{\mathcal{R}} \neq \emptyset$, the family (3.1) is said to be $\mathcal{R}$-consistent.

Let $G \in \mathcal{M}^{mn\times mn}$, $G = [g_{ij}] \in (P + Q)\{1\}$, and $|G| = [|g_{ij}|]$. Let $S_G \in \mathcal{M}^{2mn\times 2mn}$ be given by (2.2). The next theorem presents a necessary and sufficient condition for the $\mathcal{R}$-consistency of (3.1).

Theorem 3.1. *Let $A \in \mathcal{M}^{n\times n}$, $B \in \mathcal{M}^{m\times m}$ be the coefficient matrices of the FSME (2.3) for a given $n \times m$ matrix of fuzzy numbers $\tilde{C}$. Let $P = I_m \otimes A$ and $Q = B^T \otimes I_n$ be the coefficient matrices of the system (2.4) associated to the FSME (2.3), for given $\tilde{V}_{\tilde{C}} = \mathrm{vec}(\tilde{C}) \in \mathcal{V}_{mn}$. Let $X^* = S_G V_{\tilde{C}}$, where $G \in (P + Q)\{1\}$, and $\mathsf{R} = V_{\tilde{C}} - (S_P + S_Q)X^*$.*

The associated family of linear systems (3.1) for the system (2.4) is $\mathcal{R}$-consistent iff there exist $\mathsf{L} = \begin{bmatrix} \Lambda \\ -\Lambda \end{bmatrix}$, where $\Lambda = (\lambda_1(\alpha), \ldots, \lambda_{mn}(\alpha))^T$, such that $(S_P + S_Q)\mathsf{L} = \mathsf{O}$, and $\mathsf{T} = \begin{bmatrix} \Theta \\ \Theta \end{bmatrix}$, where $\Theta = (\theta_1(\alpha), \ldots, \theta_{mn}(\alpha))^T$, such that $(S_P + S_Q)\mathsf{T} = \mathsf{R}$, and $X^* + \frac{1}{2}\mathsf{L} + \mathsf{T} \in \mathcal{F}^R$.

Moreover,

$$\mathcal{ASS}^{\mathcal{R}} = \left\{ V_{\tilde{X}} \in \mathcal{F}^R \,|\, V_{\tilde{X}} = X^* + \frac{1}{2}\mathsf{L} + \mathsf{T},\ (S_P + S_Q)\mathsf{L} = \mathsf{O},\ (S_P + S_Q)\mathsf{T} = \mathsf{R} \right\}.$$

Proof. The proof is analogous to the proof of Theorem 1 in [1]. □

Based on Theorem 3.1, we propose a straightforward method for obtaining the general algebraic solution of the fuzzy Sylvester matrix equation:

Algorithm

Step 1. Transform the given FSME (2.3) into the equation (2.4) and compute $X^* = S_G V_{\tilde{C}}$, where $G \in (P + Q)\{1\}$.

Step 2. Compute a $mn \times 1$ functional vector Λ such that $(P + Q)\Lambda = \mathsf{O}$.

Step 3. For X^* obtained by Step 1, compute a $mn \times 1$ functional vector W such that $W = \underline{V_{\tilde{C}}} - [P^+ + Q^+\ P^- + Q^-]X^*$.

Step 4. If it exists, compute a $mn \times 1$ functional vector Θ such that $(|P| + |Q|)\Theta = W$, where W is obtained by Step 3, else, FSME has no solution.

Step 5. If it exists, compute the representative solution $V_{\tilde{X}} = X^* + \frac{1}{2}\mathsf{L} + \mathsf{T}$ of (3.1), else, FSME has no solution.

Step 6. By inverse operation of the vec operation (which is defined in Definition 2.1), we transform $V_{\tilde{X}}$ into matrix $\tilde{X}$, and we get the general fuzzy solutions of (2.3).

Notice that from all determined L and T by Step 2 and Step 4, we only consider such that for each $\alpha \in [0, 1]$ it holds $\Theta \leq \frac{\overline{X^*} - \underline{X^*}}{2}$, and all components of $\underline{X^*} + \frac{1}{2}\Lambda + \Theta$ are non-decreasing and left-continuous functions on the unite

interval, whereas all components of $\overline{X}^* + \frac{1}{2}\Lambda - \Theta$ are non-increasing and left-continuous functions. Obviously, for such Λ and Θ, the family of intervals $\{[\underline{x}_i^*(\alpha)+\frac{1}{2}\lambda_i(\alpha)+\theta_i(\alpha), \overline{x}_i^*(\alpha)+\frac{1}{2}\lambda_i(\alpha)-\theta_i(\alpha)] \,|\, \alpha \in [0,1]\}$, for all i, determines α-cuts of $\tilde{V}_{\tilde{X}}$, which is a solution of the system (2.4).

Example 3.2. Consider the fuzzy Sylvester matrix equation $A\tilde{X} + \tilde{X}B = \tilde{C}$:

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} + \begin{bmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \tilde{C},$$

where

$$\tilde{C} = \begin{bmatrix} (-2+2\alpha, 2-2\alpha) & (\frac{3}{2}+3\alpha, 6-\frac{3}{2}\alpha) \\ (-1+3\alpha, 5-3\alpha) & (\frac{5}{2}\alpha, 5-\frac{5}{2}\alpha) \end{bmatrix},$$

for any $\alpha \in [0,1]$. According to (2.4), the given FSME is equivalent to the following equation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} \\ \tilde{x}_{21} \\ \tilde{x}_{12} \\ \tilde{x}_{22} \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \tilde{x}_{11} \\ \tilde{x}_{21} \\ \tilde{x}_{12} \\ \tilde{x}_{22} \end{bmatrix} = \begin{bmatrix} \tilde{c}_{11} \\ \tilde{c}_{21} \\ \tilde{c}_{12} \\ \tilde{c}_{22} \end{bmatrix}.$$

The one of $\{1\}$-inverses of the matrix $P + Q = I_2 \otimes A + B^T \otimes I_2$ is in the next form:

$$G = (P+Q)^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{2}{5} \end{bmatrix}.$$

According to Algorithm, Step 1, we have:

$$X^* = S_G V_{\tilde{C}} = (0,\ -1+3\alpha,\ 1+2\alpha,\ \alpha,\ 0,\ -5+3\alpha,\ -4+\alpha,\ -2+\alpha)^T.$$

Step 2: $\Lambda = \big(f(\alpha),\, 0,\, 0,\, 0\big)^T$ is a solution vector of $(P+Q)\Lambda = \mathsf{O}$, where $y = f(\alpha)$, $\alpha \in [0,1]$, is function on the unite interval.
Next, we have:

$$W = \underline{V_{\tilde{C}}} - [P^+ + Q^+\ P^- + Q^-]X^* = \big(-2+2\alpha,\, 6-6\alpha,\, 0,\, 0\big)^T.$$

Each system in the family of classical systems $(|P|+|Q|)\Theta = W$ has a unique solution: $\Theta = \big(-1+\alpha,\, 2-2\alpha,\, 0,\, 0\big)^T$.
Finally, the general solution of the family (3.1), $V_{\tilde{X}} \in \mathcal{ASS}^R$, is $V_{\tilde{X}} = \begin{bmatrix} \underline{V_{\tilde{X}}} & -\overline{V_{\tilde{X}}} \end{bmatrix}^T$,

$$\underline{V_{\tilde{X}}} = \underline{X^*} + \frac{1}{2}\Lambda + \Theta = \begin{bmatrix} -1+\alpha+\frac{1}{2}f(\alpha) \\ 1+\alpha \\ 1+2\alpha \\ \alpha \end{bmatrix},$$

$$\overline{V_{\tilde{X}}} = \overline{X^*} + \frac{1}{2}\Lambda - \Theta = \begin{bmatrix} 1-\alpha+\frac{1}{2}f(\alpha) \\ 3-\alpha \\ 4-\alpha \\ 2-\alpha \end{bmatrix},$$

where f is a feasible function defined on the unite interval. Let $f(\alpha) = k\alpha + n$, $k, n \in \mathbb{R}$. In order to obtain a representative vector X (all the components must be fuzzy numbers), we have $1 + \frac{k}{2} \geq 0$ and $-1 + \frac{k}{2} \leq 0$, from which we conclude that $k \in [-2, 2]$, $n \in \mathbb{R}$. For example, if we take $f(\alpha) = 2$, $\alpha \in [0, 1]$, we obtain one of the infinitely many solutions of the considered FSME:

$$\tilde{X} = \begin{bmatrix} (\alpha, 2-\alpha) & (1+2\alpha, 4-\alpha) \\ (1+\alpha, 3-\alpha) & (\alpha, 2-\alpha) \end{bmatrix}.$$

4. Conclusion

In this paper, we investigated the fuzzy Sylvester matrix equation (FSME) and proposed a new algorithm for determining its general algebraic (strong) solution. The proposed method was based on the {1}-inverse and employed the Kronecker product to transform the FSME into a dual fuzzy linear system. A necessary and sufficient condition for the $\mathcal{R}$-consistency of the associated system was established. Since every real matrix possesses at least one {1}-inverse, the proposed approach provides a reliable basis for obtaining all algebraic (strong) solutions of the FSME by means of generalized inverses, without any restrictions related to the nonpositivity (nonnegativity) or the existence of generalized inverses of the coefficient matrices involved.

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