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SECOND-ORDER MOMENT INVARIANTS WITH APPLICATIONS IN IMAGE ANALYSIS

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Abstract. In this paper, we consider three well-known second-order moment invariants, such as the first two Hu moment invariants and the first affine moment invariant. These invariants are not computationally demanding, they are simple, and easy to understand. Their dependencies and shape interpretation are also included in the paper, along with an experimental illustration of their behavior on synthetically generated shapes.

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1. Introduction

Shape, as a fundamental object property akin to color and texture, is traditionally identified as a bounded, not necessarily connected region in the plane, represented by foreground pixels in a binary digital image. For the planar shape S, the *moment of order* $p+q$, denoted as $m_{p,q}(S)$, is defined as follows

$$m_{p,q}(S) = \iint_S x^p y^q dx\, dy. \tag{1.1}$$

The zero-order moment $m_{0,0}(S)$ represents the area of the shape, while $\left(\frac{m_{1,0}(S)}{m_{0,0}(S)}, \frac{m_{0,1}(S)}{m_{0,0}(S)}\right)$ represents its centroid. Since the shape remains unchanged during translation of the object itself, central moments, denoted by $\overline{m}_{p,q}(S)$, are introduced as follows:

$$\overline{m}_{p,q}(S) = \iint_{(x,y)\in S} (x - x_c(S))^p \, (y - y_c(S))^q \, dx\, dy \tag{1.2}$$

and as such they are translation invariant. However, given that the shape of the object does not change under translation for an arbitrary vector, without

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loss of generality, we can suppose that the shape considered is translated so that its centroid coincides with the origin. In this regard, if we assume that

$$m_{1,0}(S) = 0 \text{ and } m_{0,1}(S) = 0, \tag{1.3}$$

then $\overline{m}_{p,q}(S) = m_{p,q}(S)$. Given that a shape of the object does not change under the translation transformation, this assumption is not a restriction in the shape based object analysis tasks.

Additionally, in the design tasks of shape-based object analysis tools, the requirement of invariance with respect to object scaling is particularly important, since the shape of the object does not change under such transformations. Therefore, it is useful to define an appropriate moment invariant, denoted e.g. $\mu_{p,q}(S)$, such that for an arbitrary shape S and a scaling factor $\lambda > 0$ the following equality holds:

$$\mu_{p,q}(\lambda \odot S) \;=\; \mu_{p,q}(S) \tag{1.4}$$

where $\lambda \odot S = \{(\lambda \cdot x, \lambda \cdot y) \,\big|\, (x,y) \in S\}$ denotes shape S scaled by factor $\lambda > 0$.

To design such a moment-based shape feature, notice that

$$\begin{aligned} m_{p,q}(\lambda \odot S) &= \iint\limits_{\lambda \odot S} (\lambda \cdot x)^p (\lambda \cdot y)^q d(\lambda \cdot x)\, d(\lambda \cdot x) \\ &= \lambda^{p+q+2} \cdot \iint\limits_{S} x^p y^q dx\, dy = \lambda^{p+q+2} \cdot m_{p,q}(S), \end{aligned} \tag{1.5}$$

while in the case $p = q = 0$, we get that $m_{0,0}(\lambda \odot S) = \lambda^2 \cdot m_{0,0}(S)$, respectively $\lambda = \left(\frac{m_{0,0}(\lambda \odot S)}{m_{0,0}(S)}\right)^{1/2}$. Hence, after substituting the last relation in (1.5), we obtain the following:

$$m_{p,q}(\lambda \odot S) \;=\; \left(\frac{m_{0,0}(\lambda \odot S)}{m_{0,0}(S)}\right)^{(p+q+2)/2} \cdot m_{p,q}(S) \tag{1.6}$$

i.e.,

$$\frac{m_{p,q}(\lambda \odot S)}{m_{0,0}(\lambda \odot S)^{(p+q+2)/2}} \;=\; \frac{m_{p,q}(S)}{m_{0,0}(S)^{(p+q+2)/2}} \tag{1.7}$$

Now from the latter we get that the normalized moments are defined as

$$\mu_{p,q}(S) = \frac{m_{p,q}(S)}{m_{0,0}(S)^{(p+q+2)/2}}$$

and they are invariant w.r.t. translation and scaling of the shape. Since the shape of the object does not change when scaled by an arbitrary factor, in the rest of the paper we assume that all considered shapes have unit area, i.e. $m_{0,0}(S) = 1$. Therefore, we have that $\mu_{p,q}(S) = m_{p,q}(S)$, which does not diminish the generality of the presented results.

2. Main results

For the results presented in this paper, the first two Hu moment invariants derived from second-order moments are of particular importance. From a mathematical point of view, the moment invariant $I(m_{p,q})$ refers to a moment function of $m_{p,q}$ that does not change its value under various transformations, e.g. similarity or affine ones, i.e., that satisfies the condition $I(m_{p,q}(S)) = I(m_{p,q}(T[S]))$ where $T[S]$ denotes the transformed shape of S itself. These quantities represent the two best-known and most widely used moment invariants among all seven presented in [1]. The latter follows from their low computational complexity, since only second-order moments are needed for their calculation. Thus, for a given shape S of a unit area, centered at the origin, the first two Hu moment invariants are defined as follows:

$$\mathcal{H}_1(S) = \mu_{2,0}(S) + \mu_{0,2}(S), \tag{2.1}$$

$$\mathcal{H}_2(S) = \left(\mu_{2,0}(S) - \mu_{0,2}(S)\right)^2 + 4\mu_{1,1}(S)^2. \tag{2.2}$$

In addition, we will also mention here the first affine moment invariant [2], which is defined using second-order moments as follows:

$$\mathcal{A}_1(S) \;=\; \mu_{2,0}(S) \cdot \mu_{0,2}(S) - \mu_{1,1}(S)^2. \tag{2.3}$$

Using elementary calculation, it is easy to verify that the first affine moment invariant in (2.3) can be expressed as a combination of the first two Hu moment invariants in (2.1) and (2.2), i.e. the following holds:

$$\begin{aligned}\mathcal{H}_1(S)^2 - \mathcal{H}_2(S) \;&= \left(\mu_{2,0}(S) + \mu_{0,2}(S)\right)^2 - \Big(\left(\mu_{2,0}(S) - \mu_{0,2}(S)\right)^2 + 4\mu_{1,1}(S)^2\Big)\\ &= \mu_{2,0}(S)^2 + 2\cdot\mu_{2,0}(S)\cdot\mu_{0,2}(S) + \mu_{0,2}(S)^2\\ &\quad - \big(\mu_{2,0}(S)^2 - 2\cdot\mu_{2,0}(S)\cdot\mu_{0,2}(S) + \mu_{0,2}(S)^2 + 4\mu_{1,1}(S)^2\big)\\ &= 4\cdot\big(\mu_{2,0}(S)\cdot\mu_{0,2}(S) - \mu_{1,1}(S)^2\big) = 4\cdot\mathcal{A}_1(S).\end{aligned}$$

It is worth noting that all three considered second-order moment invariants are also geometric moment invariants, since they can be derived using the corresponding geometric invariants [3]. Based on this, $\mathcal{H}_1(S)$ can be expressed as the integral of the square of the distance of all points $A(x,y)$ inside the shape S (of unit area) from its centroid (i.e., origin) $O(0,0)$:

$$\boxed{\iint\limits_{A(x,y)\in S} \left|\overrightarrow{r_A}\right|^2 dx\,dy} \;=\; \iint\limits_{(x,y)\in S} (x^2+y^2)\,dx\,dy \;=\; \boxed{\mathcal{H}_1(S)}$$

where, for simplicity, the position vector of point A is denoted by $\overrightarrow{r_A}$. Regarding $\mathcal{H}_2(S)$ and $\mathcal{A}_1(S)$, they can be expressed as a multiple integral of the corresponding geometric invariants, e.g. $\Phi(\overrightarrow{r_A},\overrightarrow{r_B})$, for all pairs of points $A(x,y)$ and $B(u,v)$ varying through S. Thus, for $\mathcal{H}_2(S)$, resp. $\mathcal{A}_1(S)$, the geometric invariant $\Phi(\overrightarrow{r_A},\overrightarrow{r_B})$ has the form $\Phi(\overrightarrow{r_A},\overrightarrow{r_B}) = \left|\overrightarrow{r_A}\right|^2 \cdot \left|\overrightarrow{r_B}\right|^2 \cdot \cos 2\angle(\overrightarrow{r_A},\overrightarrow{r_B})$, resp. $\Phi(\overrightarrow{r_A},\overrightarrow{r_B}) = 2\cdot\left(\frac{1}{2}\cdot\left|\overrightarrow{r_A}\times\overrightarrow{r_B}\right|\right)^2$. Now the equality of the considered

second-order moment invariants and the multiple integrals of the corresponding invariants can be easily shown using elementary calculus as follows:

$$\boxed{\iint\limits_{A(x,y)\in S}\ \iint\limits_{B(u,v)\in S} \left|\overrightarrow{r_A}\right|^2 \cdot \left|\overrightarrow{r_B}\right|^2 \cdot \cos 2\angle(\overrightarrow{r_A},\overrightarrow{r_B})\,dx\,dy\,du\,dv}$$

$$= \iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S} (x^2+y^2)\cdot(u^2+v^2)\cdot\cos 2\varphi\,dx\,dy\,du\,dv \text{ (for } \varphi = \angle(\overrightarrow{r_A},\overrightarrow{r_B}))$$

$$= \iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S} (x^2+y^2)\cdot(u^2+v^2)\cdot\left(2\cos^2\varphi-1\right)\,dx\,dy\,du\,dv$$

$$= \iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S} 2\cdot(x^2+y^2)\cdot(u^2+v^2)\cdot\cos^2\varphi\,dx\,dy\,du\,dv$$

$$- \iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S} (x^2+y^2)\cdot(u^2+v^2)\,dx\,dy\,du\,dv$$

$$= \iint\limits_{(x,y)\in S}\iint\limits_{(u,v)\in S} 2\cdot\left(\sqrt{x^2+y^2}\cdot\sqrt{u^2+v^2}\cdot\cos\varphi\right)^2\,dx\,dy\,du\,dv$$

$$- \iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S} (x^2u^2+x^2v^2+y^2u^2+y^2v^2)\,dx\,dy\,du\,dv$$

$$= \iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S} 2\cdot((x,y)\circ(u,v))^2\,dx\,dy\,du\,dv \text{ ($\circ$ denotes the dot product)}$$

$$- \iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S} (x^2u^2+x^2v^2+y^2u^2+y^2v^2)\,dx\,dy\,du\,dv$$

$$= \iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S} \left(2\cdot(xu+yv)^2-(x^2u^2+x^2v^2+y^2u^2+y^2v^2)\right)\,dx\,dy\,du\,dv$$

$$= \iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S} \left(x^2u^2-x^2v^2-y^2u^2+y^2v^2+4\cdot xyuv\right)\,dx\,dy\,du\,dv$$

$$= \mu_{2,0}(S)^2 - 2\cdot\mu_{2,0}(S)\cdot\mu_{0,2}(S) + \mu_{0,2}(S)^2 + 4\cdot\mu_{1,1}(S)^2$$

$$= (\mu_{2,0}(S)-\mu_{0,2}(S))^2 + 4\cdot m\mu_{1,1}(S)^2 = \boxed{\mathcal{H}_2(S)}$$

where the moments $\mu_{p,q}(S)$ comes from the equality

$$\iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S} x^p y^q u^r v^s\,dx\,dy\,du\,dv$$

$$= \iint\limits_{(x,y)\in S} x^p y^q\,dx\,dy \cdot \iint\limits_{(u,v)\in S} u^r v^s\,du\,dv = \mu_{p,q}(S)\cdot\mu_{r,s}(S), \tag{2.4}$$

while for the first affine moment invariant the following applies:

$$\boxed{\iint\limits_{A(x,y)\in S}\ \iint\limits_{B(u,v)\in S} 2\cdot\left(\frac{1}{2}\cdot\left|\overrightarrow{r_A}\times\overrightarrow{r_B}\right|\right)^2\,dx\,dy\,du\,dv}$$

$$= 2\cdot\iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S}\frac{1}{4}\cdot(x\cdot v-y\cdot u)^2\,dx\,dy\,du\,dv$$

$$= \frac{1}{2}\cdot\iint\limits_{(x,y)\in S}\ \iint\limits_{(u,v)\in S}\left(x^2\cdot v^2-2\cdot x\cdot y\cdot u\cdot v+y^2\cdot u^2\right)\,dx\,dy\,du\,dv$$

$$= \frac{1}{2}\cdot\left(\mu_{2,0}(S)\cdot\mu_{0,2}(S)-2\cdot\mu_{1,1}(S)^2+\mu_{2,0}(S)\cdot\mu_{0,2}(S)\right)$$

$$= \mu_{2,0}(S)\cdot\mu_{0,2}(S)-\mu_{1,1}(S)^2=\boxed{\mathcal{A}_1(S)}.$$

Such a geometric interpretation of the moment invariant paved the way for the definition of diverse shape-based object descriptors whose behavior can be relatively easily understood and predicted in some applications, even before a specific shape-based object analysis task is performed [4, 5].

3. Experimental illustration of the behavior of the second-order moment invariants

Here we present experimental results related to three second-order moment invariants, i.e. $\mathcal{H}_1(S), \mathcal{H}_2(S)$ and $\mathcal{A}_1(S)$, applied to synthetically generated rectangular shapes centered at the origin and of unit area, i.e., $R(t) = \{(x,y)\big|x\in[-\frac{t}{2},\frac{t}{2}], y\in[-\frac{1}{2t},\frac{1}{2t}]\}$, defined for $t>0$. Therefore, for the second-order shape moments, the following applies:

$$\mu_{1,1}(R(t)) = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}}\int\limits_{-\frac{1}{2t}}^{\frac{1}{2t}} xy\,dx\,dy = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} x\,dx\cdot\int\limits_{-\frac{1}{2t}}^{\frac{1}{2t}} y\,dy = 0$$

$$\mu_{2,0}(R(t)) = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}}\int\limits_{-\frac{1}{2t}}^{\frac{1}{2t}} x^2\,dx\,dy = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} x^2\,dx\cdot\int\limits_{-\frac{1}{2t}}^{\frac{1}{2t}} dy = 2\cdot\frac{x^3}{3}\Big|_0^{\frac{t}{2}}\cdot 2\cdot y\Big|_0^{\frac{1}{2t}}$$

$$= \frac{\cancel{2}}{3}\cdot\frac{\cancel{t}^{3}\ t^2}{\cancel{8}_{\ 4}}\cdot\cancel{2}\cdot\frac{1}{\cancel{2}\cancel{t}} = \frac{t^2}{12}$$

$$\mu_{0,2}(R(t)) = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}}\int\limits_{-\frac{1}{2t}}^{\frac{1}{2t}} y^2\,dx\,dy = \int\limits_{-\frac{t}{2}}^{\frac{t}{2}} dx\cdot\int\limits_{-\frac{1}{2t}}^{\frac{1}{2t}} y^2\,dy = 2\cdot x\Big|_0^{\frac{t}{2}}\cdot 2\cdot\frac{y^3}{3}\Big|_0^{\frac{1}{2t}}$$

$$= \cancel{2}\cdot\frac{\cancel{t}}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{8}_{\ 4}\cancel{t}^{3}\ t^2} = \frac{1}{12t^2}$$

while for the second-order moment invariants we get the following results:

$$\begin{aligned}
\mathcal{H}_1(R(t)) &= \mu_{2,0}(R(t)) + \mu_{0,2}(S) = \frac{t^2}{12} + \frac{1}{12t^2} = \frac{t^4+1}{12t^2} \\
\mathcal{H}_2(R(t)) &= \left(\mu_{2,0}(R(t)) - \mu_{0,2}(R(t))\right)^2 + 4\mu_{1,1}(R(t))^2 \\
&= \left(\frac{t^2}{12} - \frac{1}{12t^2}\right)^2 = \left(\frac{t^4-1}{12t^2}\right)^2 \\
\mathcal{A}_1(R(t)) &= \mu_{2,0}(R(t)) \cdot \mu_{0,2}(R(t)) - \mu_{1,1}(R(t))^2 = \frac{t^2}{12} \cdot \frac{1}{12t^2} = \frac{1}{144}
\end{aligned}$$

Notice that for $t = 1$, the rectangle $R(t)$ becomes a unit square, centered at the origin, with second-order moment invariants given as follows:

$$\mathcal{H}_1(R(t=1)) = \frac{1}{6}, \quad \mathcal{H}_2(R(t=1)) = 0, \quad \mathcal{A}_1(R(t=1)) = \frac{1}{144}.$$

On the other hand, in the case when $t \to \infty$, the rectangle $R(t)$ degenerates into an infinitely long horizontal strip, giving $\lim_{t\to\infty} \mu_{2,0}(R(t)) = \infty$ and $\lim_{t\to\infty} \mu_{0,2}(R(t)) = 0$, i.e. $\lim_{t\to\infty} \mathcal{H}_1(R(t)) = \infty$ and $\lim_{t\to\infty} \mathcal{H}_2(R(t)) = \infty$. Conversely, in the case when $t \to 0$, $R(t)$ becomes an infinitely long vertical strip with $\mu_{2,0}(R(t)) \to 0$ and $\mu_{0,2}(R(t)) \to \infty$, while both $\mathcal{H}_1(R(t))$ and $\mathcal{H}_2(R(t))$ remain infinite.

Interestingly, in all cases for arbitrary values of t the first affine moment invariant $\mathcal{A}_1(R(t))$ does not change, i.e. it is invariant during modification of a horizontally oriented rectangle via a square to a vertically oriented rectangle and vice versa. The latter follows from the fact that the square and the rectangle are affine-invariant shapes shapes (obtained by scaling by different factors along different axes), so the value of $\mathcal{A}_1(R(t))$ remains unchanged as expected (equal to $\frac{1}{144}$ in this case). In contrast, the values of the Hu moment invariants change as expected, given the observed modification of the rectangle is not a similarity transformation.

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