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MATHEMATICAL INSIGHT INTO THE FOURIER TRANSFORM FOR 2D IMAGE ANALYSIS[1]

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Abstract. This paper presents the mathematical foundations of the Fourier transform and its use in edge detection on digital images. Beyond a practical implementation, the focus is on providing a mathematical insight into how frequency-domain analysis relates to spatial intensity variations in images, including its role in modern applications such as super-resolution. A high-pass filter is applied in the frequency domain to isolate high-frequency components that correspond to edges. The theoretical concepts are supported by a numerical implementation of the 1D and 2D Fast Fourier Transform using the Python programming language. The results demonstrate that frequency-domain filtering provides a global and mathematically robust method for edge extraction, offering a deeper understanding of how changes in pixel intensity relate to frequency characteristics in digital images.

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1. Introduction

The Fourier transform is a fundamental tool in signal and image processing, providing a link between the spatial and frequency domains. Through the Discrete Fourier Transform (DFT) and its efficient implementation, the Fast Fourier Transform (FFT), images can be analyzed and filtered based on their spectral components [1, 2].

Edges in digital images correspond to high-frequency variations, which represent rapid transitions in intensity or color. Studies have shown that combining spatial and frequency-domain information improves the robustness of edge

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detection in complex scenes [3], while emphasizing high-frequency components enhances the extraction of fine details [4].

Beyond edge detection, frequency-domain analysis has found new applications in modern computer vision. In particular, Fourier-based techniques have contributed to super-resolution, where recovering high-frequency information leads to sharper and more detailed images [5].

This work aims to provide a concise mathematical perspective on the use of the 2D DFT in image analysis, focusing on both edge detection and resolution enhancement. The goal is to explore how classical frequency-domain techniques continue to shape modern image analysis, bridging fundamental theory with contemporary computational approaches.

2. Mathematical Foundations of the Fourier Transform

This section introduces the main mathematical concepts underlying the Fourier transform and its application to signal and image analysis.

A *signal* is a function that conveys information about a physical process over time or space. It can be either continuous (analog) or discrete (sampled, digital), with examples including audio waves, electric voltage, or images. The basic characteristics of a signal include its *amplitude* (the maximum value relative to zero, indicating signal strength), *frequency* (number of oscillations per unit time, measured in Hz) and *phase* (the wave change relative to a reference point) [6].

2.1. Discrete Fourier Transform (DFT)

The Discrete Fourier Transform converts a discrete signal from the time domain to the frequency domain.

Definition 2.1 (1D Discrete Fourier Transform [1]). Let $x[n]$ be a discrete signal of length N. Its DFT is defined as:

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-i\frac{2\pi}{N}kn}, \quad k = 0, 1, \dots, N-1$$

Figure 1 illustrates a simple one-dimensional discrete signal composed of two sinusoidal components. Figure 1a shows the discrete signal in the time domain, while Figure 1b displays the corresponding magnitude spectrum obtained via the Fast Fourier Transform (FFT). The magnitude spectrum represents the strength of each frequency component: peaks appear at 50 Hz and 120 Hz, corresponding to the signal's frequency components.

The term FFT refers to a family of efficient algorithms for implementing the discrete Fourier transform [1]. While direct computation of an N-*point* DFT requires N^2 complex multiplications, FFT methods reduce this to the order of $N \log_2 N$ complex multiplications.

Remark 2.2 ([1]). The inverse DFT reconstructs the original signal:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{i\frac{2\pi}{N}kn}$$

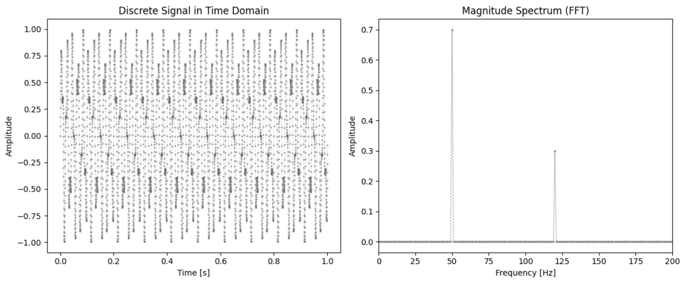

Figure 1: 1D discrete signal: (a) time domain; (b) frequency domain

2.2. 2D Discrete Fourier Transform for Images

A digital image can be interpreted as a 2D signal, i.e., a function of two discrete variables (m, n).

Definition 2.3 (2D Discrete Fourier Transform [2])**.** Let $f[m, n]$ be a 2D signal of size $M \times N$. Its 2D DFT is defined as:

$$F[u,v] = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]e^{-i2\pi\left(\frac{um}{M}+\frac{vn}{N}\right)}, \quad u = 0,\ldots,M-1,\ v = 0,\ldots,N-1$$

Here, $F[u, v]$ represents the *frequency spectrum* of the image, where each pair of coordinates (u, v) corresponds to a specific spatial frequency component. Low frequencies are concentrated near the center of the spectrum, describing smooth intensity variations, while high frequencies (located toward the edges) represent rapid intensity changes, such as edges and fine details [7].

Remark 2.4 ([2])*.* The inverse 2D DFT reconstructs the original image:

$$f[m,n] = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F[u,v]e^{i2\pi\left(\frac{um}{M}+\frac{vn}{N}\right)}$$

2.3. Frequency-Domain Edge Detection

Edges in images correspond to rapid spatial intensity variations, which are captured by high-frequency components in the Fourier domain.

Definition 2.5 (High-Pass Filter [8])**.** A high-pass filter is a frequency-domain filter that suppresses low frequencies while preserving high frequencies corresponding to image edges.

Remark 2.6 ([8])*.* Applying the high-pass filter in the frequency domain and performing the inverse 2D DFT produces an image emphasizing edges.

Remark 2.7 ([8])*.* Mathematically, the filtering operation in the frequency domain can be expressed as:

$$\tilde{F}[u,v] = F[u,v] \cdot H(u,v),$$

where $H(u,v)$ is the high-pass filter mask defined as:

$$H(u,v) = \begin{cases} 0, & (u-u_0)^2 + (v-v_0)^2 \le r^2, \\ 1, & \text{otherwise.} \end{cases}$$

Here, (u_0, v_0) denotes the center of the frequency spectrum, and r is the radius of the suppressed low-frequency region. This operation removes smooth intensity variations while retaining high-frequency components corresponding to image edges, and corresponds to the ideal high-pass filter formulation.

3. Application: Edge Detection Using FFT

The implementation of frequency-domain edge detection was carried out in Python using the `NumPy`, `Matplotlib`, and `scikit-image` libraries. A color image was first loaded and converted to grayscale, forming a two-dimensional discrete signal $f[m,n]$.

The 2D Fast Fourier Transform (FFT) was applied to map the image from the spatial domain, where pixel intensities vary with position, to the frequency domain, where the same information is represented as a combination of sinusoidal components of different frequencies. For practical filtering and visualization, the resulting frequency spectrum must be shifted (using a function `np.fft.fftshift`) to move the zero-frequency component from the corners to the center of the spectrum (see Figure 2b). In this domain, low frequencies correspond to slowly varying intensity regions (smooth areas), while high frequencies correspond to rapid intensity changes - that is, edges and fine details.

After constructing and applying the high-pass filter as defined in Section 2, the inverse FFT (`np.fft.ifft2`) was used to reconstruct the filtered image back into the spatial domain. Since only high-frequency components remain, the resulting image highlights regions with strong intensity gradients - the edges.

This implementation demonstrates the connection between the Fourier transform and edge detection: frequency-domain filtering emphasizes high-frequency components globally, which mathematically correspond to the spatial derivatives approximated by methods like Sobel or Canny edge detector.

Figure 2 shows the complete workflow of frequency-domain edge detection.

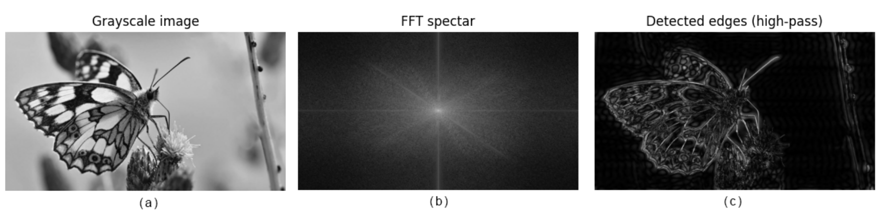

Figure 2: Edge detection: (a) original image in grayscale; (b) logarithmic magnitude spectrum; (c) edge-detected image after high-pass filtering

4. Modern Applications of the Fourier Transform

Beyond the classical applications of the Fourier transform (e.g., filtering for edge detection), recent research has explored its integration with modern machine learning and image enhancement techniques. For example, in the study [9] it was demonstrated that applying FFT-based frequency-domain conversion prior to convolutional processing significantly improves classification accuracy.

In another line of work, super-resolution methods exploit the recovery of high-frequency components via Fourier encodings to enhance image resolution. As a practical example, the method from the work [5] was implemented in Python, producing the upsampled image shown in Figure 3. The implementation was performed on a grayscale butterfly image, upsampled by a factor of four in the frequency domain. The resulting red-tinted image highlights recovered fine details and improved sharpness, while the extracted high-frequency components emphasize edges, showing how Fourier-based enhancement preserves structural boundaries and contrast.

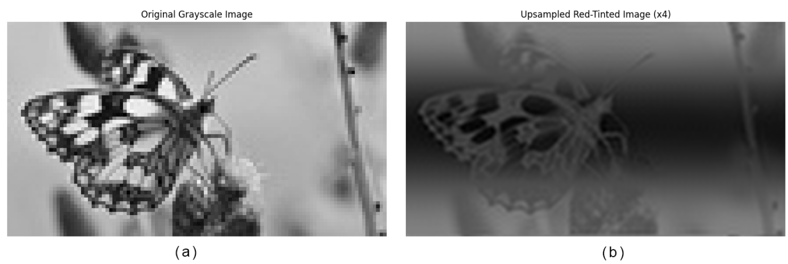

Figure 3: Super-resolution result: (a) original image; (b) upsampled image

These developments underscore that the Fourier transform remains not only a fundamental mathematical tool, but also a gateway to hybrid methods combining signal processing and deep learning. While the present study focuses on the use of the 2D DFT and high-pass filtering for edge detection, the above works point to promising future directions such as frequency-domain neural networks and super-resolution pipelines.

5. Conclusion

This paper has presented the mathematical foundations of the 2D Discrete Fourier Transform and its practical applications in image analysis. This study shows how frequency-domain filtering can be used to extract edges by isolating high-frequency components, highlighting the connection between spatial intensity variations and their spectral representation. Additionally, it discusses the role of the Fourier transform in modern image enhancement, such as super-resolution, demonstrating its continued relevance in both classical and contemporary applications. Overall, the presented work provides a clear mathematical perspective that can serve as a basis for further exploration of frequency-domain methods in image processing and hybrid approaches combining Fourier analysis with modern computational techniques.

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