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ORIENTATION-ASSISTED NULL SPACE SEARCH FOR TOMOGRAPHY RECONSTRUCTION

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Abstract. This paper contains a review of Tomography Reconstruction based on Null Space Search, with an accent on application and using orientation as a prior information.

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1. Introduction

Tomography is a branch of image processing concerned with reconstructing unknown images from given projection data. Mathematically, an image can be represented as a function whose range may be continuous or discrete. In Computerized Tomography (CT), this image function has a continuous range. Discrete Tomography (DT) is a subfield of tomography in which the image function takes values from a finite and discrete set. When this range includes only a few predefined intensity levels, the method is called Multi-Level Tomography. A special case of this is Binary Tomography, where the image contains only two intensity values—usually 0 and 1.

Numerous methods have been proposed in the literature to solve the tomography reconstruction problem. Among the most well-known iterative algorithms are the Algebraic Reconstruction Technique (ART)[[8]], the Simultaneous Iterative Reconstruction Technique (SIRT)[[7]], the Discrete Algebraic Reconstruction Technique (DART)[[9]], the Spectral Projected Gradient (SPG) method[[10]], and the Difference of Convex functions (DC) algorithm. A common characteristic of these approaches is that they iteratively adjust the pixel intensities to reduce the projection error—i.e., they minimize the difference between the measured and reconstructed projections during convergence.

This limitation motivated us to develop a new reconstruction technique in which the projection error remains unchanged throughout the iterative process. Since the projection data are typically the most reliable information about the unknown image, maintaining a minimal projection error is crucial. The proposed method addresses this issue by ensuring that, during reconstruction, the projection error theoretically remains constant at zero or in practical applications, very close to zero.

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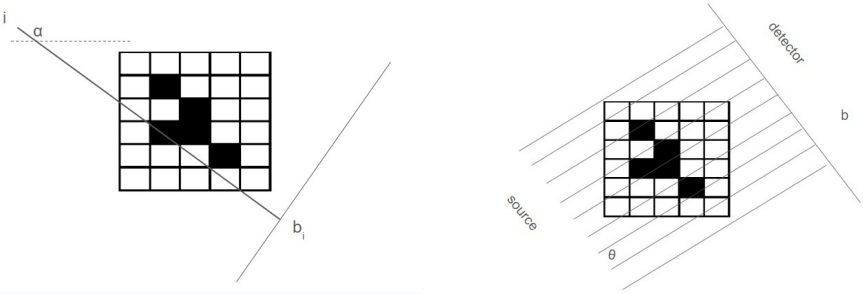

Figure 1: Transmision tomography reconstruction model from two different projection angles. Left image represent i-th projection ray from the projection angle α passing through the object, and it's detected projection value b_i. Right image represent set of parallel projection rays from the projection angle θ.

2. Definitions

Mathematically, the problem of tomography reconstruction may be formulated by the following system of linear equations

$$A\,u = b, \quad A \in \mathbb{R}^{m\times n}, \;\; u \in \mathbb{R}^n, \;\; b \in \mathbb{R}^m, \tag{2.1}$$

where u represents the unknown image that should be reconstructed. In the case of binary tomography, the components of vector u have only two different values, usually 0 and 1. Rows of matrix A (projection matrix) hold information about the length of the projection ray that passes through the pixel. The assumption is that each pixel is represented as a square with unit side length, see Figure 1. The figure represents the projection value calculation for a given image from one projection direction denoted by angle α. The another projection direction is obtained by rotation source-detector system around of the center of the circle. Each projection direction contributes a new parallel set of the projection rays. Detected projection values are placed in the projection vector b.

In the reconstruction problem, both the matrix A and the projection vector b are given, as a calculated or measured data. The task is to determine the unknown image u. The system (2.1) is often underdetermined ($m << n$), and consequently, in general case, we can count on an infinite number of possible solutions.

3. Tomography Reconstruction Based on Null Space Search (NSST)

The main idea uses the fact that any solution of the projection linear system of equations (2.1)

$$A\,u = b,$$

can be represented as a sum of one particular solution u_p ($A\,u_p = b$) and an appropriate vector belongs to the null space of the matrix A, defined by

$$\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid A\,x = 0\}.$$

It is known the set of all solutions can be represented as

$$u_p + z, \quad \text{where} \quad z \in \mathcal{N}(A).$$

We will denote a basis of the vector space $\mathcal{N}(A)$ by the set of vectors $\{\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_k\}$, where $k = n-m$. Each solution of the system (2.1) may be represented in the following form

$$\begin{gathered} w(\alpha) = u_p + \alpha_1\mathbf{b}_1 + \alpha_2\mathbf{b}_2 + \ldots + \alpha_k\mathbf{b}_k, \\ \text{where } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{R}^k \text{ and } \mathbf{b}_i \in \mathbb{R}^n \text{ for all } i = 1, \ldots, k. \end{gathered} \tag{3.1}$$

Further, it is necessary to choose coefficients $\alpha_1, \alpha_2, \ldots \alpha_k$ in α in a such way that the obtained solution $w(\alpha) \in \mathbb{R}^n$ lies in a predefined area, for example in the hyper cube $[0,1]^n$. There, we get an optimization problem that has to be solved. The whole procedure is described in detail in the paper [[1], [3]]. The described NSST method is significantly practical and applicable for low-dose and limited-angle computed tomography (CT). Because the method can reconstruct images from a small number of projections, it is valuable in scenarios where radiation exposure must be minimized (e.g., pediatric imaging). It can also reconstruct images when full rotational access around the patient is not possible, such as in mobile CT systems or dental imaging. In industrial nondestructive testing, components are often too large or heavy to rotate freely, leading to incomplete projection data. The approach is particularly beneficial for composite materials or additive manufacturing quality control, where internal structures may be known to possess certain regularities or symmetries that can be incorporated as prior information.

4. Orientation

In this section, we want to use orientation of the object to improve quality of Tomography Reconstruction. First, we will give a short reminder of object orientation. The most usual definition refers on axis of the least second moment of inertia.This is the line which minimizes the integral of the squared distances of the points (whose belong to the object) to the line. Mathematicaly, we want to optimize the integral

$$\iint_S r^2(x, y, \alpha, \rho)dxdy \tag{4.1}$$

where $r(x, y, \alpha, \rho)$ is the distance from the point (x, y) to the line $X \sin\alpha - Y\cos\alpha = \rho$. As a result of the optimization, it is obtained formula for the shape orientation θ given by

$$\mathrm{tg}(2\theta) = \frac{2\overline{m}_{1,1}(S)}{\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S)} \tag{4.2}$$

where $\overline{m}_{i,j}(S)$ is central geometric moment i, j [[2]] for the planar shape S. On the other hand, we can observe orientation as a line u on which there is the greatest dispersion of the projected points of the object, ie. this one who takes a maximum of information of object:

$$\max_u \sum_i \left(x_i^T \cdot u\right)^2 u^T u = 1,$$

where x_i is point that belong to the object. Therefore, we have the optimization problem with constraint whose solution satisfies a matrix equation. The eigenvector of that matrix represents shape orientation. Also, it is the principal component 1 (PC1) whose carry maximum of information of object.

We will use orientation as an angle of projection and also the angle which is normal to orientation [[2]] to improve reconstruction.

We used original phantom images, Fig 2.

Figure 2: Original phantom images used in our experimental work

Here, it is presented a short experimental work with previous phantoms. We tested if using orientation will improve reconstruction process. The common feature of these phantoms is object elongation, ie. object with high elongation are suitable for this type of improving reconstruction process. In all cases, regardless of the projective direction projection error (PE) has a very small value, approximately $9.8 \cdot 10^{-8}$. That is expected because it's the main advantage of NSST. Reconstruction error $E_R(u^r) = \sum_{i=1}^{n} |u_i^r - u_i^*|$, is significantly smaller in case we use shape orientation and angle normal to orientation. For phantom PH1 reconstruction error is 537 in case using horizontal and vertical projection direction, and 59 when using orientation, see Figure 3. First column is for central reconstruction, second for binary reconstructions and third for presentation of misclassified pixel. For phantom PH2, situation is 740 to 42 in favor of NSST, Figure 4. The best situation is absolute reconstruction, i.e. when error is equal to zero, but we are satisfied when it is as much as possible close to zero.

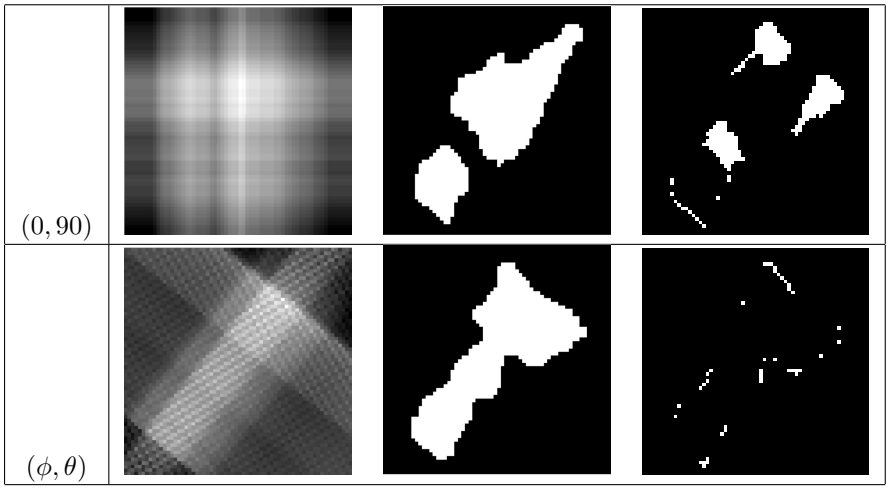

Figure 3: Comparison of central reconstruction, binary reconstruction and difference between original and binary reconstruction for PH1 in case using $(0, 90)$ or (ϕ, θ) where θ is angle of orientation and ϕ is normal to orientation.

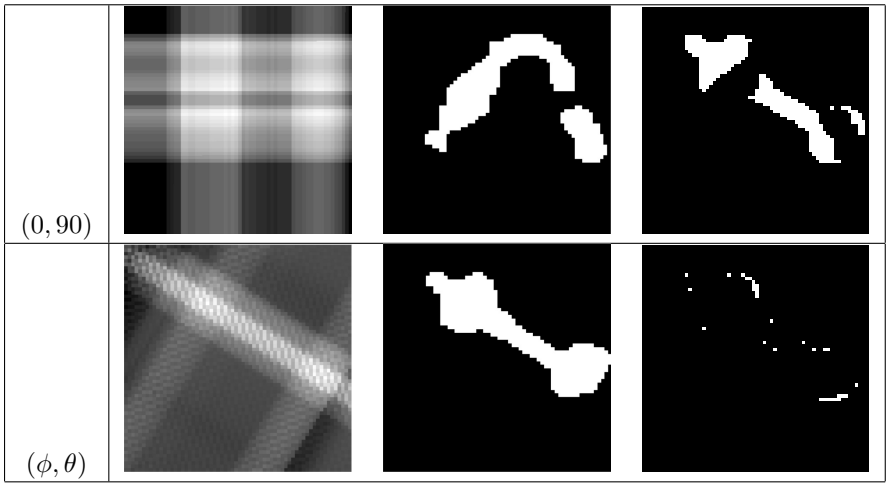

Figure 4: Comparison of central reconstruction, binary reconstruction and difference between original and binary reconstruction for PH2 in case using $(0, 90)$ or (ϕ, θ) where θ is angle of orientation and ϕ is normal to orientation.

5. Conclusion

The Tomography Reconstruction based on Null Space Search is briefly described with the emphasis on application. Also, we used an orientation to improve reconstruction. There are several ways to continue the research, both by using a prior information and by changing the focus on multi-level images.

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